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MEDITATIONES
ALGEBRAICÆ.

AB ^K
EDVARDO WARING, M.D.
MAG. COLL. CANTAB. SOC.
MATHESEOS PROFESSORE LUCASIANO,
REGIÆ SOCIETATIS,
ET
BONONIENSIS SCIENTIARUM ACADEMIÆ SOCIO.

CANTABRIGIÆ,
TYPIS ACADEMICIS EXCUDEBAT J. ARCHDEACON.
Veneunt apud J. WOODYER, Cantabrigiæ; J. BEECROFT, T. PAYNE,
et T. CADELL, Londini; et D. PRINCE, Oxon.
M.DCC.LXX.

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MEDITATIONES HASCE ALGEBRAICAS

OMNI CUM OBSERVANTIA

D. D. D.

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MEDITATIONES HASCE ALGEBRAICAS

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D.D.D.

EDVARDUS WARING.

P R Æ F A T I O.

DE incrementis iis, quæ gradatim res ceperit algebraica, narrationem hic contexui brevem, ut sua inventoribus deferatur gloria; atque ut iis simul, qui progressus artium investigant curiosius, aliquâ sit ex parte satisfactum. Ex historiis clar. virorum Wallisii & Monteculi quædam mutuatus sum, quorum alter Harriotto nostrati nimis favet, alter quidem gallicis scriptoribus, sed humanum est sic errare. Controversiæ serram de procurfibus, quos in hac disciplinâ fecerint antiqui, reciprocare vix operæ pretium erit: Vulgares certe arithmeticiæ operationes, æquationumque tum simplicium tum quadraticarum resolutiones probe calluerunt. Diophantus Alexandrinus, qui floruit circa annum domini 360, algebraistarum, quorum opera supersunt, est antiquissimus. Multas æquationes, quæ duas incognitas quantitates involvunt, ita reduxit, ut quantitates innotescerent incognitæ; artesque quibus utitur in valoribus rationalibus inveniendis adeo concinnæ & elegantes sunt, ut a recentioribus raro superentur. Figuræ numerariæ 0, 1, 2, 3, &c. ab Indis inventæ ad Arabes primo transferunt, & deinde inter annos 1000 & 1100 vel forsan autea in usum apud Europæos venerunt. Arithmetica decimalis secundum Wallisium ab Arzachel & Regiomontano circiter annum 1400 inchoata est, & a Purblech quoque, referente Monteculo: Radium quidem in partes 60000000 diviserunt; sed fractiones invenisse decimales vix dici possunt. Ad umbilicum fere perducta est decimalium methodus a Petro Ramo in arithmetica suâ circa annum 1560 editâ, ubi docet in extrahendâ radice quadraticâ cubicâve (si quid residuum sit post integrorum operationem) tot punctationes cipharum subjungere, quot opus videbitur, & operationem proseguiri ut in integris, quo habeatur radix in integris cum adjunctis partibus decimalibus. Idem jam ante factum esse a nostrate Gulielmo Buclæo Collegii Regalis apud Cantabrigienses alumno asserit Wallisius, cujus arithmeticam memorativam (versu conscriptam) subjunctam vidit Setoni lo-

gica editæ Cantabrigiæ anno 1631: Buclæus autem mortuus est circa annum 1550 (regnante Edvardo sexto, qui obiit anno 1553).

Mohammed Ben Musa & Thebit Ben Corah *algebraistarum* apud Arabes vetustissimi sunt, quorum primus, auctore Cardano, æquationum quadraticarum resolutionem invenit: Dicitur quoque Omar Ben Ibrahim cubicas reduxisse æquationes: Arabica autem MSS. non satis scrutati sunt recentiores. Leonardum de Pisa collocat Vossius circa annum 1400 aut citius, eumque dicit (ex Blancano) primum fuisse ex hodiernis, qui de *algebrâ* scripsit. Primus autem, qui tractatum de *algebra* typis exhibuit, erat Lucas de Burgo; omnia, quousque æquationes pertingunt quadraticæ, tradidit, earumque duas esse radices ostendit: Æquationis cubicæ resolutionem Scipioni Ferreo attribuit Cardanus, methodum vero suam studiose celavit Ferreus; eandem interim suo Marte invenit Tartalea, & Cardano roganti, & se arcanum fore pollicenti, communicavit; fidem vero fefellit Cardanus, methodumque Tartaleæ auctam tamen atque promotam edidit; secundum æquationis cubicæ terminum abstulit Cardanus, casumque irreducibilem tres vel negativas vel affirmativas habere radices, ostendit. Æquationis biquadraticæ resolutionem invenit Ludovicus Ferrarius, Bombelliumque suam methodum ex Ludovico Ferrario haussisse dicit Wallisius in epistolâ ad Morlandum Cap. 53. *algeb.* — Quoad notationem Buteo in ejus *logisticâ* anno 1559 editâ (sub finem libri tertii in suâ regulâ quantitatis dictâ) varias quæsitæ quantitates literis A, B, C, &c. designavit. Vieta primum literas pro cognitis quantitatibus substituit, atque coefficientem secundi æquationis termini summam esse radicum, & similiter omnium coefficientium constitutionem docuit, etiamque radices datæ æquationis per datas quantitates auxit & diminuit, multiplicavit & divisit. De quantitate binomiali ad potestatem quamlibet eveclâ verba facit, dicitque coefficientem termini secundi unum fore e numeris triangularibus, coefficientem tertii termini unum fore e numeris pyramidalibus, & sic deinceps. Quantitates aa, aaa, &c. aliquando a², a³, &c. pro quantitatibus Aq, Ac, &c. substituit Harriottus, æquationem integram ad unam partem revocat, & nihilo facit æqualem, & exinde veram originem æquationum superiorum ex compositione latera-

lium

lium & coefficientium constitutionem ostendit: convertit etiam radices omnes negativas in affirmativas, & vice versâ, mutatis locorum parium signis. Hunc insecutus est Cartesius, ultimas literas alphabetæ x, y, z, &c. pro incognitis quantitatibus, primas vero a, b, c, &c. pro cognitis adhibet; tot esse radices affirmativas, quot sint mutationes signorum de + in — & — in +, cæteras vero negativas esse, si modo omnes æquationis radices essent possibiles, dixit Cartesius, cujus regulæ demonstrationem primus tradidit Gua; novam methodum resolvendi æquationem biquadraticam invenit, & quædam quoque de divisoribus integris datæ æquationis eruendis, & de numeris amicabilibus adjecit Cartesius. Methodum extrahendi radicem quamlibet (n) e quantitate binomiali $a + \sqrt[n]{b}$, si modo radix possit exprimi per quantitatem $\frac{x + \sqrt[n]{y}}{\sqrt[n]{Q}}$, dedit Schootenius; quam qui-

dem magis concinnam reddidit Neutonus, & deinde quæsitam radicem e regulâ prædictâ non semper provenire invenit, novamque addidit Eulerus: datæ æquationis simplices divisores invenire substituendo — 1, 0, 1, &c. pro incognitâ quantitate docuit Jacobus a Waessanaer, secundas differentias quadratorum esse æquales, tertias cuborum, & sic deinceps, invenit Collinsius, & hinc methodum Waessaneri ad divisores quadraticos, cubicos, &c. inveniendos promovit Neutonus: Modum inveniendi utrum ulla æquatio rationalem divisorem recipiat, necne, tradidit Huddenius: Ex æquatione qualibet tollit radicalia & terminos successivos æquationis datæ per arithmeticam seriem multiplicat, & exinde, ubi radices datæ æquationis fuerint æquales, invenit, vel potius secundum Erasmum Bartholinum æquationem invenit, cujus radices sunt limites inter radices æquationis datæ: Duarum æquationum reductionem ad unam, ita ut incognita quantitas exterminetur, docuit Erasmus Bartholinus. Fractionem, uti loquuntur, continuam & ejusdem legem primus tradidit Vice comes Brounker; dato uno valore integro quantitatis x vel y in æquatione $ax^2 + 1 = y^2$, modum plures inveniendi deduxit. Hanc methodum ad æquationem $ax^2 + b = y^2$ promovit Eulerus. Methodum reducendi fractiones & rationes ad minores terminos, servato quam potest proxime valore, dedit Wallisius; metho-

methodumque etiam approximandi ad valores integros incognitarum quantitatum in æquatione $ax + 1 = v^2$ & similibus æquationibus; eadem fere fecit Pellius. Numerorum primorum quasdam elegantes proprietates tradidit Fermatius. Regulam dedit Neutonus, ex quâ inveniri potest summa datæ æquationis radicum, summa quadratorum e singulis radicibus, cuborum, &c. invenit etiam regulam, ex quâ sæpe cognosci potest, quot datæ æquationis radices sint possibiles, & quot impossibiles; & quasdam æquationes $(2n)$ dimensionum in æquationes (n) dimensionum reduxit; radicem quadraticam quantitatis $a + \sqrt{b} + \sqrt{c} + \sqrt{d} + \&c.$ si modo radix exprimi possit per quantitates ejusdem generis, extraxit; & theorema binomiale, si non primus invenerit, promovit tamen, atque ad extractions radicum aliosque usus primus omnium adhibuit; sequentem quoque dedit regulam de electione terminorum ad calculum ineundum, si duæ incognitæ quantitates similiter involvantur in datis æquationibus, tum neutrum adhibere convenit, sed eorum vice tertium eligere, qui similem utrique relationem gerit: In resolutione problematis hoc artificio prius usus est Schootenius & forsan alii, sed nemo ante Neutonom diserte tradidit regulam, cujus rei ratio a nobis primum in pag. 53 & 54 miscell. anal. prolata fuit.

Fractionem $\frac{1}{x^n - px^{n-1} + \&c.}$ in fractiones $\frac{p}{x-\alpha} + \frac{q}{x-\beta} + \&c.$ primum distinxit Leibnitzius. Resolutionem æquationis hujusce formulæ $x^n - nx^{n-2} + n \times \frac{n-3}{2} x^{n-4} - \&c. = 0$, & reductionem æquationis recurrentis $x^{2n} + px^{2n-1} \dots px + 1 = 0$ ad æquationem $x^n - ax^{n-1} + \&c. = 0$ reperiit Moivræus. Æquationem $x^n - px^{n-1} + \&c. = 0$ ad æquationem $x^m - ax^{m-1} + \&c. = 0$ ope æquationis $n \cdot \frac{n-1}{2} \dots \frac{n-m+1}{m}$ dimensionum deprimi posse ostendit Le Seur, atque inde demonstravit æquationem $x^{4n+2} + \&c. = 0$ unum quadraticum divisorem possibilem ad minimum habere; & ex iisdem sane principiis demonstravit Eulerus quantitatem $x^n - px^{n-1} + \&c.$ constare e simplicibus vel quadraticis divisoribus in sese ductis: non nisi post edita mea miscell. anal. librum Dⁱ Le Seur videram, nec Euleri methodum a meâ tamen multo

multo discrepantem, nisi post solutionem meam impressam obtinere contigit. Duas æquationes in unam, ita ut exterminaretur incognita quantitas novâ methodo reduxit Cramerus. Resolutionem biquadraticæ æquationis & reductionem duarum æquationum in unam, ita ut incognita quantitas exterminaretur alio modo docuit Eulerus. Elegantes quasdam proprietates numerorum integrorum atque primorum, in capite quinto hujusce libri contentorum, & propositiones consimiles a Fermatio sine probationibus editas, demonstratas dedit Eulerus: Regulas quasdam de numeris impossibilium radicum inveniendis dederunt Mac Laurin, Campbell & Gua. Dominus Clairaut in algebrâ suâ 1760 editâ methodum inveniendi numerum radicum impossibilium in biquadraticâ æquatione contentarum tradidit: Diversam de eâdem re methodum anno 1757 ad Societatem Regiam ipse transmiseram, & impressa est anno 1760.

In Actis Parisiensibus anni 1764 æquationes quasdam invenit Bezout, quarum resolutiones sunt $x = \sqrt[n]{p^m} + b \sqrt[n]{p^{n-m}}$ & $x = \sqrt[n]{p} + \sqrt[n]{p^2} \dots + \sqrt[n]{p^{n-1}}$: Resolutionem vero generaliore $x = a \sqrt[n]{p} + b \sqrt[n]{p^2} + c \sqrt[n]{p^3} + \text{Ec.}$ in meis miscell. analy. anno 1760 dederam.

Post tot atque tantorum virorum in re algebraicâ labores, est quod miretur forsan aliquis, me alium de eodem argumento edere tractatum, nec chartæ jam parcere perituræ: Delictum tamen meum, si quod fuerit, hæc erunt, quæ minuant: 1^{mo}, librum, qui omnia recentiorum scriptorum inventa complectatur, extare nullum; deinde, me quoque quædam, quæ luce forsan non indigna erunt, invenisse; sed de iis judicet eruditus lector: Officio certe meo fideliter inservire, scientiæque fines, quâ potui, promovere, in votis erat. Quid in hoc libro, de rebus potius quam de verbis laborans, præstiti, paucis accipe.

In primo capite continetur nova methodus inveniendi summam quadratorum, cuborum, &c. e singulis datæ æquationis radicibus: Consequitur methodus inveniendi summam omnium quantitatum hujusce generis $\alpha^a \beta^b \gamma^c$ &c. $+ \alpha^b \beta^a \gamma^c$ &c. $+ \text{Ec.}$ ubi $\alpha, \beta, \gamma, \text{Ec.}$ sint radices datæ æquationis, cujus series mirâ gaudet simplicitate: Consecratur methodus inveniendi sum-

nam quarumcunque functionum algebraicarum e radicibus compositarum: Et datâ unâ, duabus vel pluribus æquationibus docetur methodus inveniendi æquationem, cujus radices sint quæcunque algebraica radicum datarum æquationum functio, satis facile e principiis maxime notis deducenda; sed, quantum scio, a nemine in usum ante me recepta. In exemplo primo invenitur æquatio, cujus radices sint (m) potestates datæ æquationis radicum, & exinde facile deleri possint surdæ quantitates e datâ æquatione; & nova aliquot adjiciuntur exempla.

In capite secundo probatur omnem æquationem, cujus radices sint omnes impossibiles fore aggregatum e diversis quadratis; nova etiam quædam theoremata traduntur, e quibus deduci possint Neutoni, Mac Laurini, & consimiles de numero radicum impossibilium inveniendo regulæ, atque Cartesii etiam regula de numero affirmativarum & negativarum datæ æquationis radicum: Dantur etiam generalia principia numerum impossibilium radicum, affirmativarum & negativarum, eruendi; & exinde deducuntur regulæ, e quibus dici potest, quot radices impossibiles, affirmativas & negativas habent æquationes biquadraticæ & formulæ hujusce $x^n + Ax^m + B = 0$; & numerus quoque radicum impossibilium in æquatione quinque dimensionum contentarum invenitur: Datur etiam methodus inveniendi, utrum ulla habet radices impossibiles data æquatio, necne: Et datâ æquatione duas vel plures incognitas quantitates habente, traditur methodus inveniendi, utrum hæ incognitæ quantitates possibiles recipiant correspondentes valores, necne. In scholio tot vel plures esse mutationes signorum in datâ æquatione de + in - & - in +, quot affirmativæ radices; tot vel plures continuos progressus de + in + & - in -, quot negativæ radices; demonstratur.

In capite tertio continetur generalis methodus inveniendi illas datæ æquationis radices, quæ datam inter se habent relationem; datur etiam æquatio, quæ habet infinitas radices; eas vero cognitæ: Quædam etiam adjiciuntur nova de resolutionibus cubicæ & biquadraticæ æquationis per methodos Ferrei & Ferrarii. Alia etiam continetur resolutio biquadraticæ æquationis; dantur etiam methodi inveniendi æquationes, quas deprimere vel resolvere liceat; & quantitate datâ, quæ simplicium terminorum juxta dimensiones literæ x progredientium seriem exprimit, invenitur quantitas alternorum

ternorum seriei terminorum, vel terminorum binis vel ternis, &c. intervallis a se invicem distantium summæ, æqualis; hanc methodum cum scriptis aliis ad Regiam Societatem anno 1757 transmissi: In scholio asseritur, quod approximatio radices datæ æquationis e notis regulis inventa haud pendeat e ratione, quam quantitas assumpta pro vero valore habet ad verum valorem, sed ex approximatione ad unum valorem proprii quam ad reliquas.

In capite quarto exponitur modus, per quem in vulgari methodi exterminandi incognitas quantitates irrepant in æquationem resultantem radices, quæ in datis haud inveniuntur, & quæ & quot sint illæ radices. Sint duæ æquationes n & m dimensionum duas incognitas quantitates habentes, reducantur hæ duæ æquationes in unam, ita ut exterminetur altera incognita quantitas & ostenditur in quibus casibus quædam radices nibilo vel inter se fiant æquales vel prorsus evanescant, & exinde constant dimensiones, ad quas resultans æquatio assurgit: Et datâ æquatione unam vel duas incognitas quantitates habente, traditur methodus inveniendi constitutionem coefficientium, & exinde inveniri possint quocunque volueris æquationes duas incognitas quantitates habentes, quarum correspondentes valores incognitarum quantitatum sint datæ quantitates: Instituuntur modi inveniendi duas æquationes duas vel plures incognitas quantitates habentes, quas resolvere vel deprimere licet: Quædam etiam adjiciuntur de summis diversarum functionum e radicibus incognitarum quantitatum in duabus vel pluribus æquationibus contentarum; de æquationibus duas vel plures incognitas quantitates similiter involventibus; & in scholio quædam de approximationibus vel ad possibiles vel ad impossibiles radices in pluribus æquationibus tot incognitas quantitates habentibus consulantur.

In capite quinto continentur maxime recentiorum inventa, quædam vero nova de integris divisoribus datarum æquationum tot incognitas quantitates habentium, & de resolvendis & reducendis æquationibus radicum quadraticarum, cubicarum, &c. ope; de serie continuæ fractionis inveniendâ, de integris valoribus datæ æquationis duas vel plures incognitas quantitates habentis; etiamque de pluribus æquationibus in unam reducendis, ita ut exterminentur incognitæ quantitates, & si incognitæ quantitates resultantium æquationum sint rationales, exterminatæ etiam erunt rationales:

tionales: Paucula etiam nova adjiciuntur de primis, integris & exponentialibus numeris: Subjicitur etiam, quod si modo sint duæ æquationes duas incognitas quantitates x & y habentes, tum correspondentes valores quantitatum x & y semper eandem involvant irrationalitatem, ni duo vel plures valores quantitatum x vel y sint inter se æquales; traditur postremo proprietas maxime elegans primorum numerorum ab amicissimo & in omni mathe- seos parte versatissimo viro Joanne Wilson Armigero inventa atque mihi communicata. Gratias quoque, quas debeo, maximas referam Joanni Law, A.M. Coll. Chris. (viro egregiæ in mathematicâ re peritiæ), qui pro humanitate suâ exemplar revisit, & plurima correxit.

Multa sparsim insuper adjeci, & ut labores hosce meos benigne excipiant, & defectus in calculis adeo intricatis rerum algebraicarum cultores suppleant, ex animo oro & exopto.

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MEDI-

MEDITATIONES ALGEBRAICAE.

C A P U T I.

DE methodo inveniendi æquationem, cujus radices sint quæcunque algebraica radicum datæ æquationis vel datarum æquationum functio.

P R O B. I.

Sint $\alpha, \beta, \gamma, \delta, \epsilon$, &c. Radices datæ æquationis

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + vx^{n-6} - wx^{n-7} + zx^{n-8} \&c. = 0.$$

invenire summam $\alpha^m + \beta^m + \gamma^m + \delta^m + \epsilon^m + \&c.$

Summa quæfita erit

[illegible]

Duplex est lex, quam observat hæc Series; altera, quæ contenta literalia; altera, quæ coefficientes seu uncias respicit.

Contentorum literalium lex est; ut datæ æquationis coefficientium exponentes in singulis contentis eidem potestati quantitatis (p) annectendis eandem conficiant summam, quæ quidem summa est u , si modo p^{m-u} fit potestas quantitatis p , cui annectuntur contenta hæc literalia.

Per datæ æquationis coefficientium exponentes intelligo numeros designantes earum distantias a primo æquationis termino; sic expo-

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nens coefficientis q , est 2; coefficientis r , 3; coefficientis s , 4; coefficientis t , 5; et sic deinceps; quo fit, ut exponens cujuslibet potestatis coefficientis q , qualis q^μ , sit 2μ ; exponens cujuslibet potestatis coefficientis r , qualis r^ν , sit 3ν ; exponens cujuslibet potestatis coefficientis s , qualis s^π , sit 4π ; exponens cujuslibet potestatis coefficientis t , qualis t^ρ , sit 5ρ ; &c.

Cuncta insuper contenta literalia, quæ summam u possunt conficere, potestati p^{m-u} sunt semper adjungenda.

Coefficientium seu unciarum hæc est lex. Cujuscunque literalis contenti $p^{m-u} q^\mu r^\nu s^\pi t^\rho v^\sigma w^\tau$ &c. Coefficientis erit fractio, cujus numerator est

$$m \times m - u + 1 \times m - u + 2 \times m - u + 3 \times m - u + 4 \times m - u + 5 \times m - u + 6 \times \&c.$$

in quo tot factores erunt, quot unitates in summâ indicum $\mu, \nu, \pi, \rho, \sigma, \tau$, &c. denominator vero erit contentum ortum e multiplicatione serierum

$$1 \times 2 \times 3 \times 4 \times \&c. \quad 1 \times 2 \times 3 \times 4 \times \&c. \quad 1 \times 2 \times 3 \times 4 \times \&c. \quad 1 \times 2 \times 3 \times 4 \times \&c.$$

quarum prima continet tot terminos, quot sunt unitates in μ ; secunda, quot sunt in ν ; tertia, quot sunt in π ; quarta, quot sunt in ρ ; &c.

Coefficientis literalis contenti $p^{m-u} q^\mu r^\nu s^\pi t^\rho$ &c. signum erit affirmativum, vel negativum, (modo u sit par numerus) prout summa $\mu + \nu + \pi + \rho + \&c.$ sit par vel impar: quod si u sit impar numerus, signum erit affirmativum, vel negativum, prout impar vel par fuerit summa $\mu + \nu + \pi + \rho + \&c.$

D E M O N S T R A T I O.

* " Sit $p=a$, $p^2-2q=b$, $pb-qa+3r=c$, $pc-qb+ra-4s=d$,
 " $pd-qc+rb-sa+5t=e$, $pe-qd+rc-sb+ta-6v=f$, & sic
 " deinceps. Et erit a summa radicum, b summa quadratorum ex
 " singulis radicibus, c summa cuborum, d summa quadrato-quadrato-
 " torum, e summa quadrato-cuborum, f summa cubo-cuborum, et
 " sic in reliquis."

Ex hac regulâ satis facile colligitur coefficientem literalis contenti $p^{m-u} q^\mu r^\nu s^\pi t^\rho$ &c. in summa $\alpha^m + \beta^m + \gamma^m + \delta^m + \&c.$ invenien-

dâ, æqualem esse aggregato coefficientium literalium contentorum p^{m-u-1} .

$p^{m-u-1} q^\mu r^\nu s^\pi t^\rho \&c.$, $p^{m-u} q^{\mu-1} r^\nu s^\pi t^\rho \&c.$, $p^{m-u} q^\mu r^{\nu-1} s^\pi t^\rho \&c.$,
 $p^{m-u} q^\mu r^\nu s^{\pi-1} t^\rho \&c.$, $p^{m-u} q^\mu r^\nu s^\pi t^{\rho-1} \&c.$, in summis respectivis
 $\alpha^{m-1} + \beta^{m-1} + \gamma^{m-1} + \delta^{m-1} + \&c.$, $\alpha^{m-2} + \beta^{m-2} + \gamma^{m-2} + \delta^{m-2} + \&c.$, $\alpha^{m-3} +$
 $\beta^{m-3} + \gamma^{m-3} + \delta^{m-3} + \&c.$, $\alpha^{m-4} + \beta^{m-4} + \gamma^{m-4} + \delta^{m-4} + \&c.$, $\alpha^{m-5} + \beta^{m-5} +$
 $\gamma^{m-5} + \delta^{m-5} + \&c.$, $\&c.$

Hæ autem coefficientes respective erunt

$$\begin{array}{l} \frac{m-1 \times m-u \times \frac{m-u+1 \times m-u+2 \times m-u+3 \dots \times m-u+\mu+\nu+\pi+\rho+\&c.-2}{1.2.3\dots\mu \times 1.2.3\dots\nu \times 1.2.3\dots\pi \times 1.2.3\dots\rho \times \&c.}}{\mu \times m-2 \times \frac{m-u+1 \times m-u+2 \times m-u+3 \dots \times m-u+\mu+\nu+\pi+\rho+\&c.-2}{1 \times 2.3\dots\mu \times 1.2.3\dots\nu \times 1.2.3\dots\pi \times 1.2.3\dots\rho \times \&c.}} \\ \frac{\nu \times m-3 \times \frac{m-u+1 \times m-u+2 \times m-u+3 \dots \times m-u+\mu+\nu+\pi+\rho+\&c.-2}{1.2.3\dots\mu \times 1.2.3\dots\nu \times 1.2.3\dots\pi \times 1.2.3\dots\rho \times \&c.}}{\pi \times m-4 \times \frac{m-u+1 \times m-u+2 \times m-u+3 \times \dots \times m-u+\mu+\nu+\pi+\rho+\&c.-2}{1.2.3\dots\mu \times 1.2.3\dots\nu \times 1.2.3\dots\pi \times 1.2.3\dots\rho \times \&c.}} \\ \frac{\rho \times m-5 \times \frac{m-u+1 \times m-u+2 \times m-u+3 \times \dots \times m-u+\mu+\nu+\pi+\rho+\&c.-2}{1.2.3\dots\mu \times 1.2.3\dots\nu \times 1.2.3\dots\pi \times 1.2.3\dots\rho \times \&c.}}{\&c.} \end{array}$$

Et aggregatum harum unciam erit

$$\begin{array}{l} m^2-u m+u \\ -1 \quad -2\mu \\ +\mu \quad -3\nu \\ +\nu \quad -4\pi \\ +\pi \quad -5\rho \\ +\rho \quad \&c. \\ +\&c. \end{array} \times \frac{m-u+1 \times m-u+2 \times m-u+3 \times \dots \times m-u+\mu+\nu+\pi+\rho+\&c.-2}{1.2.3\dots\mu \times 1.2.3\dots\nu \times 1.2.3\dots\pi \times 1.2.3\dots\rho \times \&c.}$$

In primo hujus aggregati factore pro u substituatür ejus valor
 $(2\mu + 3\nu + 4\pi + 5\rho + \&c.)$; et fiet aggregatum

$$m \times \frac{m-u+\mu+\nu+\pi+\rho+\&c.-1 \times \frac{m-u+1 \times m-u+2 \times m-u+3 \times \dots \times m-u+\mu+\nu+\pi+\rho+\&c.-2}{1.2.3\dots\mu \times 1.2.3\dots\nu \times 1.2.3\dots\pi \times 1.2.3\dots\rho \times \&c.}}{1.2.3\dots\mu \times 1.2.3\dots\nu \times 1.2.3\dots\pi \times 1.2.3\dots\rho \times \&c.}$$

Hoc vero aggregatum est coefficientis seu uncia, quæ in datâ serie pro
 uncia literalis contenti $p^{m-u} q^\mu r^\nu s^\pi t^\rho \&c.$ in summâ $\alpha^m + \beta^m +$
 $\gamma^m + \delta^m + \&c.$ substituitur; & consequenter, cum series prædicta
 inferiores summâs $\alpha^{m-1} + \beta^{m-1} + \gamma^{m-1} + \delta^{m-1} + \&c.$, $\alpha^{m-2} + \beta^{m-2} +$
 $\gamma^{m-2} + \delta^{m-2} + \&c.$, $\alpha^{m-3} + \beta^{m-3} + \gamma^{m-3} + \delta^{m-3} + \&c.$, $\alpha^{m-4} +$
 $\beta^{m-4} + \gamma^{m-4} + \delta^{m-4} + \&c.$, recte ostendat, summam quoque
 proxime superiorem $\alpha^m + \beta^m + \gamma^m + \delta^m + \&c.$ vere exprimet.

Ex. Sit æquatio $x^4 - 2x^3 - 7x^2 + 20x - 12 = 0$. In hoc ex-
 emplo p, q, r, s, t , sunt respective 2, -7, -20, -12, 0. Quare

summa quadratorum e singulis radicibus ($p^2 - 2q$) est $2^2 + 2 \times 7 = 18$; cuborum summa ($p^3 - 3pq + 3r$) $2^3 + 3 \times 2 \times 7 - 3 \times 20 = -10$; quadrato-quadratorum summa ($p^4 - 4qp^2 + 4rp - 4s + 2q^2$) $2^4 + 4 \times 7 \times 2^2 - 4 \times 20 \times 2 + 4 \times 12 + 2 \times 7^2 = 114$. Quadrato-cuborum summa ($p^5 - 5qp^3 + 5rp^2 - 5s + 5q^2$) $2^5 + 5 \times 7 \times 2^3 - 5 \times 20 \times 2^2 + 5 \times 12 \times 2 + 5 \times 7^2 \times 2 - 5 \times 7 \times 20 = -178$.

Non abs re forsan erit observare seriem

$$\left(\begin{array}{l} p^m - mqp^{m-2} + mrp^{m-3} - m \\ + m \times \frac{m-3}{2} q^2 \end{array} \right\} p^{m-4} + m \times \frac{m-4}{2} qr \left\} p^{m-5} - \&c. \right)$$

in infinitum excurrentem, in alteram seriem sui similem

$$\left(\begin{array}{l} p^n - nqp^{n-2} + nrp^{n-3} - n \\ + n \times \frac{n-3}{2} q^2 \end{array} \right\} p^{n-4} + n \times \frac{n-4}{2} qr \left\} p^{n-5} - \&c. \right)$$

&c. in infinitum) ductam, seriem infinitam

$$\left(\begin{array}{l} p^{n+m} - \overline{n+m} qp^{n+m-2} - \overline{n+m} mrp^{n+m-3} + \overline{n+m} \\ - \overline{n+m} \times \frac{n+m-3}{2} q^2 \end{array} \right\} p^{n+m-4} + \overline{n+m} \times \frac{n+m-4}{2} qr \left\} p^{n+m-5} + \&c. \right)$$

$$p^{n+m-4} - \overline{n+m} \times \frac{n+m-4}{2} qr \left\} p^{n+m-5} + \&c. \right)$$

eandem prorsus legem quoad terminos servantem efficere.

PROB. II.

Invenire aggregatum e pluribus hujuscemodi seriebus confectum.

Addantur simul serierum termini, & conficitur problema.

Ex. Sit a summa radicum, b summa quadratorum e singulis radicibus, c summa cuborum, d summa quadrato-quadratorum, e summa quadrato-cuborum, f summa cubo-cuborum, &c.

Invenire summam (m) terminorum ab initio hujusce seriei

$$a + b + c + d + e + \&c.$$

Per problema præcedens

$$a =$$

$$a = p$$

$$b = p^2 - 2q$$

$$c = p^3 - 3qp + 3r$$

$$d = p^4 - 4qp^2 + 4rp - 4s + 2q^2$$

$$e = p^5 - 5qp^3 + 5rp^2 - 5sp + 5q^2p + 5t - 5qr$$

&c.

&c.

&c.

&c.

&c.

&c.

$$p^m - mqp^{m-2} + mrp^{m-3} - ms p^{m-4} + m \times \frac{m-3}{2} q^2 p^{m-4} + m t p^{m-5} - m \times m - 4qr p^{m-5} - \&c.$$

summa autem primæ columnæ

$$p + p^2 + p^3 + p^4 + p^5 \dots p^m = \frac{p^{m+1} - p}{p - 1}$$

secundæ vero

$$2q + 3qp + 4qp^2 + 5qp^3 \dots mqp^{m-2} = \frac{2 - p - m + 1p^{m-1} + mp^m}{1 - p^2} \times q$$

tertiæ

$$3r + 4rp + 5rp^2 \dots mrp^{m-3} = \frac{3 - 2p - m + 1p^{m-2} + mp^{m-1}}{1 - p^2} \times r$$

quartæ

$$4s + 5sp + 6sp^2 \dots ms p^{m-4} = \frac{4 - 3p - m + 1p^{m-3} + mp^{m-2}}{1 - p^2} \times s$$

quintæ

$$2q^2 + 5q^2p + 9q^2p^2 \dots m \times \frac{m-3}{2} p^{m-4} = \frac{4 - 2p - m + 1 \times m - 2p^{m-3} + 2m^2 - 4m - 4p^{m-2} - m \times m - 3p^{m-1}}{2 \times 1 - p^3} \times q^2$$

&c.

= &c.

et consequenter summa quæsitæ erit

$$\frac{p^{m+1} - p}{p - 1} - \frac{2 - p - m + 1p^{m-1} + mp^m}{1 - p^2} \times q + \frac{3 - 2p - m + 1p^{m-2} + mp^{m-1}}{1 - p^2} \times r$$

$$- \frac{4 - 3p - m + 1p^{m-3} + mp^{m-2}}{1 - p^2} \times s + \frac{5 - 4p - m + 1p^{m-4} + mp^{m-3}}{1 - p^2} \times t$$

$$+ \frac{4 - 2p - m + 1 \times m - 2 \times p^{m-3} + 2m^2 - 4m - 4p^{m-2} - m \times m - 3p^{m-1}}{2 \times 1 - p^3} \times q^2 - \frac{5 - 3p - m + 1 \times m - 3p^{m-4} + 2m^2 - 6m - 5p^{m-3} - m \times m - 4p^{m-2}}{1 - p^3} \times q^2$$

qr

&c.

Eadem

Eadem erunt literalia hujusce seriei contenta ac præcedentis seriei, abjectâ quantitate p , ejusque potestatibus.

Regula, quam observant coefficientes, hæc est; scribatur $u = 2\mu + 3\nu + 4\pi + 5\rho + 6\sigma + \&c.$ & $l = 1 + \mu + \nu + \pi + \rho + \sigma + \&c.$ & coefficientis literalis contenti $q\mu r\nu s\pi t\rho u\sigma \&c.$ erit æqualis contento $1 \times 2 \times 3 \times 4 \times 5 \dots l-2$ in fractionem multiplicato, cujus denominator est

$1-p^1 \times 1 \times 2 \times 3 \times \dots \mu \times 1.2.3 \dots \nu \times 1 \times 2.3 \dots \pi \times 1.2.3 \dots \rho \times 1.2.3 \dots \sigma \times \&c.$

Numerator vero

$$\begin{aligned}
 & u - u - l + 1 \times p \\
 & - \frac{m+1}{1} \times \frac{m-u+2}{1} \times \frac{m-u+3}{2} \times \frac{m-u+4}{3} \times \frac{m-u+5}{4} \times \frac{m-u+6}{5} \dots \frac{m-u+l-1}{l-2} p^{m-u+1} \\
 & + u \times \frac{m-u+1}{1} \times \frac{m-u+3}{1} \times \frac{m-u+4}{2} \times \frac{m-u+5}{3} \times \frac{m-u+6}{4} \times \frac{m-u+7}{5} \times \frac{m-u+8}{6} \dots \frac{m-u+l}{l-2} \left. \vphantom{\frac{m-u+1}{1}} \right\} p^{m-1} \\
 & - u - l + 1 \times \frac{m-u}{1} \times \frac{m-u+2}{1} \times \frac{m-u+3}{2} \times \frac{m-u+4}{3} \times \frac{m-u+5}{4} \times \frac{m-u+6}{5} \times \frac{m-u+7}{6} \dots \frac{m-u+l-1}{l-2} \left. \vphantom{\frac{m-u+1}{1}} \right\} p^{m-1} \\
 & - u \times \frac{m-u+1}{1} \times \frac{m-u+2}{2} \times \frac{m-u+4}{1} \times \frac{m-u+5}{2} \times \frac{m-u+6}{3} \times \frac{m-u+7}{4} \times \frac{m-u+8}{5} \dots \frac{m-u+l}{l-3} \left. \vphantom{\frac{m-u+1}{1}} \right\} p^{m-1} \\
 & + u - l + 1 \times \frac{m-u}{1} \times \frac{m-u+1}{2} \times \frac{m-u+3}{1} \times \frac{m-u+4}{2} \times \frac{m-u+5}{3} \times \frac{m-u+6}{4} \times \frac{m-u+7}{5} \dots \frac{m-u+l-1}{l-3} \left. \vphantom{\frac{m-u+1}{1}} \right\} p^{m-1} \\
 & + u \times \frac{m-u+1}{1} \times \frac{m-u+2}{2} \times \frac{m-u+3}{3} \times \frac{m-u+5}{1} \times \frac{m-u+6}{2} \times \frac{m-u+7}{3} \times \frac{m-u+8}{4} \dots \frac{m-u+l}{l-4} \left. \vphantom{\frac{m-u+1}{1}} \right\} p^{m-1} \\
 & - u - l + 1 \times \frac{m-u}{1} \times \frac{m-u+1}{2} \times \frac{m-u+2}{3} \times \frac{m-u+4}{1} \times \frac{m-u+5}{2} \times \frac{m-u+6}{3} \times \frac{m-u+7}{4} \dots \frac{m-u+l-1}{l-4} \left. \vphantom{\frac{m-u+1}{1}} \right\} p^{m-1} \\
 & - u \times \frac{m-u+1}{1} \times \frac{m-u+2}{2} \times \frac{m-u+3}{3} \times \frac{m-u+4}{4} \times \frac{m-u+6}{1} \times \frac{m-u+7}{2} \times \frac{m-u+8}{3} \dots \frac{m-u+l}{l-5} \left. \vphantom{\frac{m-u+1}{1}} \right\} p^{m-1} \\
 & + u - l + 1 \times \frac{m-u}{1} \times \frac{m-u+1}{2} \times \frac{m-u+2}{3} \times \frac{m-u+3}{4} \times \frac{m-u+5}{1} \times \frac{m-u+6}{2} \times \frac{m-u+7}{3} \dots \frac{m-u+l-1}{l-5} \left. \vphantom{\frac{m-u+1}{1}} \right\} p^{m-1} \\
 & + u \times \frac{m-u+1}{1} \times \frac{m-u+2}{2} \times \frac{m-u+3}{3} \times \frac{m-u+4}{4} \times \frac{m-u+5}{5} \times \frac{m-u+7}{1} \times \frac{m-u+8}{2} \dots \frac{m-u+l}{l-6} \left. \vphantom{\frac{m-u+1}{1}} \right\} p^{m-1} \\
 & - u - l + 1 \times \frac{m-u}{1} \times \frac{m-u+1}{2} \times \frac{m-u+2}{3} \times \frac{m-u+3}{4} \times \frac{m-u+4}{5} \times \frac{m-u+6}{1} \times \frac{m-u+7}{2} \dots \frac{m-u+l-1}{l-6} \left. \vphantom{\frac{m-u+1}{1}} \right\} p^{m-1} \\
 & - u \times \frac{m-u+1}{1} \times \frac{m-u+2}{2} \times \frac{m-u+3}{3} \times \frac{m-u+4}{4} \times \frac{m-u+5}{5} \times \frac{m-u+6}{6} \times \frac{m-u+8}{1} \dots \frac{m-u+l}{l-7} \left. \vphantom{\frac{m-u+1}{1}} \right\} p^{m-1} \\
 & + u - l + 1 \times \frac{m-u}{1} \times \frac{m-u+1}{2} \times \frac{m-u+2}{3} \times \frac{m-u+3}{4} \times \frac{m-u+4}{5} \times \frac{m-u+5}{6} \times \frac{m-u+7}{1} \dots \frac{m-u+l-1}{l-7} \left. \vphantom{\frac{m-u+1}{1}} \right\} p^{m-1}
 \end{aligned}$$

& sic deinceps ad penultimum terminum $p^{m-u+l-1}$; ultimus autem terminus erit

$m \times$

$$m \times \frac{m-u+1}{1} \times \frac{m-u+2}{2} \times \frac{m-u+3}{3} \times \frac{m-u+4}{4} \dots \frac{m-u+l-2}{l-2} \times p^{m-u+1}.$$

E methodo inveniendi summam $(m-u+1)$ terminorum seriei emergentis per divisionem quantitatis $u-u-l+1 \times p$ per $1-p$, facile erui & demonstrari potest hoc problema.

Ex. Datâ æquatione $x^4-3x^3+x^2+3x-2=0$. invenire summam $a+b+c+d$.

Pro p, q, r, s, m , scribantur respective 3, 1, -3, -2, 4. Et erit summa

$$\begin{aligned} a+b+c+d &= \frac{p-p^5}{1-p} (120) - q \times \frac{2-p-5p^3+4p^4}{1-p} (-47) + r \times \\ &\frac{3-2p-5p^2+4p^3}{1-p} (-45) - s \times \frac{4-3p-5p^2+4p^3}{1-p} (8) = 88. \\ &+ q^2 \times \frac{4-2p-10p^2+12p^3-4p^4}{2 \times 1-p^2} (2). \end{aligned}$$

Cor. Si $p=1$, tum $\frac{p-p^{m+1}}{1-p}$ fit (m) , & $q \times \frac{2-p-m+1p^m}{1-p^2} + mp^m$ fit

$$\left(q \times \frac{m+2}{2} \times \frac{m-1}{2} \right), \&c.$$

Summa quæsitæ est

$$\begin{aligned} m-m+2 \times \frac{m-1}{2} q + m+3 \times \frac{m-2}{2} r - m+4 \times \frac{m-3}{2} s \\ + \frac{1}{2} \times 2m+4 \times \frac{m-3}{2} \times \frac{m-2}{3} q^2 - \&c. \end{aligned}$$

coefficientis cujuscunque literalis contenti $q^u r^v s^w t^x$ &c. hujusce seriei æqualis est fractioni

$$\frac{m-u+1 \times m-u+2 \times m-u+3 \dots m-u+l-1}{l-1 \times m+u \times l \times l-1 \times 1.2.3 \dots \mu \times 1.2.3 \dots \nu \times 1.2.3 \dots \pi \times 1.2.3 \dots \rho \times \&c.}$$

PROB. III.

Sint $\alpha, \beta, \gamma, \delta, \epsilon$, &c. radices datæ æquationis

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0,$$

Et sint $a, b, c, d, e, f, \&c.$ dati indices, quorum numerus sit m , Et m aut æqualis aut minor erit quam n .

Invenire summam omnium hujusmodi quantitatum.

$$\alpha^a \beta^b \gamma^c \delta^d \epsilon^e \&c. + \alpha^b \beta^a \gamma^c \delta^d \epsilon^e \&c. + \alpha^c \beta^a \gamma^b \delta^d \epsilon^e \&c. + \\ \alpha^d \beta^a \gamma^b \delta^c \epsilon^e \&c. + \alpha^e \beta^a \gamma^b \delta^c \epsilon^d \&c. + \alpha^a \beta^c \gamma^b \delta^d \epsilon^e \&c. + \&c.$$

in quibus numerus factorum est (m).

Scribantur $Sa, Sb, Sc, \&c.$ pro summis respectivis $a, b, c, \&c.$ potestatum e singulis radicibus, i. e.

$$Sa = \alpha^a + \beta^a + \gamma^a + \delta^a + \epsilon^a + \&c.$$

$$Sb = \alpha^b + \beta^b + \gamma^b + \delta^b + \epsilon^b + \&c.$$

$$Sc = \alpha^c + \beta^c + \gamma^c + \delta^c + \epsilon^c + \&c.$$

$$Sd = \alpha^d + \beta^d + \gamma^d + \delta^d + \epsilon^d + \&c.$$

$$Se = \alpha^e + \beta^e + \gamma^e + \delta^e + \epsilon^e + \&c.$$

$$\&c. = \&c. \quad \&c.$$

$$Sa + b = \alpha^{a+b} + \beta^{a+b} + \gamma^{a+b} + \delta^{a+b} + \epsilon^{a+b} + \&c.$$

$$Sa + c = \alpha^{a+c} + \beta^{a+c} + \gamma^{a+c} + \delta^{a+c} + \epsilon^{a+c} + \&c.$$

$$Sb + c = \alpha^{b+c} + \beta^{b+c} + \gamma^{b+c} + \delta^{b+c} + \epsilon^{b+c} + \&c.$$

$$Sa + d = \alpha^{a+d} + \beta^{a+d} + \gamma^{a+d} + \delta^{a+d} + \epsilon^{a+d} + \&c.$$

$$\&c. = \&c. \quad \&c.$$

$$Sa + b + c = \alpha^{a+b+c} + \beta^{a+b+c} + \gamma^{a+b+c} + \delta^{a+b+c} + \epsilon^{a+b+c} + \&c.$$

$$Sa + b + d = \alpha^{a+b+d} + \beta^{a+b+d} + \gamma^{a+b+d} + \delta^{a+b+d} + \epsilon^{a+b+d} + \&c.$$

$$Sa + b + e = \alpha^{a+b+e} + \beta^{a+b+e} + \gamma^{a+b+e} + \delta^{a+b+e} + \epsilon^{a+b+e} + \&c.$$

$$Sa + c + d = \alpha^{a+c+d} + \beta^{a+c+d} + \gamma^{a+c+d} + \delta^{a+c+d} + \epsilon^{a+c+d} + \&c.$$

$$Sb + c + d = \alpha^{b+c+d} + \beta^{b+c+d} + \gamma^{b+c+d} + \delta^{b+c+d} + \epsilon^{b+c+d} + \&c.$$

$$\&c. = \&c. \quad \&c. \quad \&c.$$

$$Sa + b + c + d = \alpha^{a+b+c+d} + \beta^{a+b+c+d} + \gamma^{a+b+c+d} + \delta^{a+b+c+d} + \&c.$$

$$Sa + b + c + e = \alpha^{a+b+c+e} + \beta^{a+b+c+e} + \gamma^{a+b+c+e} + \delta^{a+b+c+e} + \&c.$$

$$Sa + b + d + e = \alpha^{a+b+d+e} + \beta^{a+b+d+e} + \gamma^{a+b+d+e} + \delta^{a+b+d+e} + \&c.$$

$$Sb + c + d + e = \alpha^{b+c+d+e} + \beta^{b+c+d+e} + \gamma^{b+c+d+e} + \delta^{b+c+d+e} + \&c.$$

$$\&c. = \&c. \quad \&c. \quad \&c.$$

$$Sa + b + c + d + e = \alpha^{a+b+c+d+e} + \beta^{a+b+c+d+e} + \gamma^{a+b+c+d+e} + \&c.$$

$$\&c. = \&c.$$

& sic deinceps.

Scribatur etiam

$$A = Sa \times Sb \times Sc \times Sd \times Se \times \&c.$$

$$B = \frac{Sa+b \times A}{Sa \times Sb} + \frac{Sa+c \times A}{Sa \times Sc} + \frac{Sb+c \times A}{Sb \times Sc} + \frac{Sa+d \times A}{Sa \times Sd} + \&c.$$

$$C = \frac{Sa+b+c \times A}{Sa \times Sb \times Sc} + \frac{Sa+b+d \times A}{Sa \times Sb \times Sd} + \frac{Sa+c+d \times A}{Sa \times Sc \times Sd} + \frac{Sb+c+d \times A}{Sb \times Sc \times Sd} + \frac{Sa+b+e \times A}{Sa \times Sb \times Se} + \&c.$$

$$D = \frac{Sa+b+c+d \times A}{Sa \times Sb \times Sc \times Sd} + \frac{Sa+b+c+e \times A}{Sa \times Sb \times Sc \times Se} + \frac{Sa+c+d+e \times A}{Sa \times Sc \times Sd \times Se} + \frac{Sb+c+d+e \times A}{Sb \times Sc \times Sd \times Se} + \&c.$$

$$BB = \frac{Sa+b \times Sc+d \times A}{Sa \times Sb \times Sc \times Sd} + \frac{Sa+c \times Sb+d \times A}{Sa \times Sc \times Sb \times Sd} + \frac{Sa+e \times Sb+c \times A}{Sa \times Se \times Sb \times Sc} + \&c.$$

$$E = \frac{Sa+b+c+d+e \times A}{Sa \times Sb \times Sc \times Sd \times Se} + \frac{Sa+b+c+d+f \times A}{Sa \times Sb \times Sc \times Sd \times Sf} + \frac{Sa+b+c+e+f \times A}{Sa \times Sb \times Sc \times Se \times Sf} + \&c.$$

$$BC = \frac{Sa+b \times Sc+d+e \times A}{Sa \times Sb \times Sc \times Sd \times Se} + \frac{Sa+c \times Sb+d+e \times A}{Sa \times Sc \times Sb \times Sd \times Se} + \frac{Sa+d \times Sb+c+e \times A}{Sa \times Sd \times Sb \times Sc \times Se} + \&c.$$

$$F = \frac{Sa+b+c+d+e+f \times A}{Sa \times Sb \times Sc \times Sd \times Se \times Sf} + \&c.$$

$$BD = \frac{Sa+b \times Sc+d+e+f \times A}{Sa \times Sb \times Sc \times Sd \times Se \times Sf} + \frac{Sa+c \times Sb+d+e+f \times A}{Sa \times Sc \times Sb \times Sd \times Se \times Sf} + \&c.$$

$$CC = \frac{Sa+b+c \times Sd+e+f \times A}{Sa \times Sb \times Sc \times Sd \times Se \times Sf} + \frac{Sa+b+d \times Sc+e+f \times A}{Sa \times Sb \times Sd \times Sc \times Se \times Sf} + \&c.$$

$$BBB = \frac{Sa+b \times Sc+d \times Se+f \times A}{Sa \times Sb \times Sc \times Sd \times Se \times Sf} + \frac{Sa+c \times Sb+d \times Se+f \times A}{Sa \times Sc \times Sb \times Sd \times Se \times Sf} + \&c.$$

& sic deinceps.

Observandum est terminos prædictos Sa , Sb , Sc , &c. $Sa+b$, $Sa+c$, $Sb+c$, &c. $Sa+b+c$, $Sa+b+d$, &c. designare quantitates, quæ sibimetipsis per notam æqualitatis adjungantur.

Idem intelligendum est de quantitatibus BB , BC , BD , CC , BBB , &c.

Summa quæsitæ est

$$\begin{aligned} A - B + 1.2C - 1.2.3D + 1.2.3.4E - 1.2.3.4.5F \\ + 1.1BB - 1.1.2BC + 1.1.2.3BD \\ + 1.2.1.2CC \\ - 1.1.1BBB \\ + 1.2.3.4.5.6G - \&c. \\ - 1.1.2.3.4BE + \&c. \\ - 1.2.1.2.3CD \\ + 1.1.1.2BBC \quad \&c. \end{aligned}$$

B

Literalium

Literalium quantitatum hæc est lex. Exponentes literalium quantitatum terminorum infra se positorum eandem conficiunt summam, quæ quidem summa erit successive 1, 2, 3, 4, 5, 6, &c.

Per exponentes literarum intelligo numeros designantes sedes illas, quas in serie naturali literarum singulæ literæ occupant; sic exponens literæ *A* est 1; literæ *B*, 2; literæ *C*, 3; & sic deinceps: quo fit, ut exponens literalis quantitatis *BBB...CCC...DDD...EEE...FFF...GGG...&c.*; in quâ tot sunt literæ *B, C, D, E, F, G, &c.* contentæ, quot unitates in $\mu, \nu, \pi, \rho, \sigma, \tau, \&c.$ respective; fiat $2\mu + 3\nu + 4\pi + 5\rho + 6\sigma + 7\tau + \&c.$

Cunctæ insuper literales quantitates, quæ summas successivas (1, 2, 3, 4, 5, &c.) possunt efficere, semper sunt adjungendæ.

Coefficiens literalis quantitatis *BBB...CCC...DDD...EEE...FFF...GGG...&c.* in quâ tot sunt literæ *B*, quot unitates in μ ; tot sunt literæ *C*, quot unitates in ν ; tot sunt literæ *D, E, F, G, &c.* quot unitates in $\pi, \rho, \sigma, \tau, \&c.$ respective; æqualis est contento $1^\mu \times 1^\nu \times 2^\pi \times 1^\rho \times 2^\sigma \times 3^\tau \times 1^\sigma \times 2^\rho \times 3^\sigma \times 4^\tau \times 1^\tau \times 2^\sigma \times 3^\tau \times 4^\tau \times 5^\tau \times 6^\tau \times \&c.$

Quod si sint (*m*) indices quantitates æquales per (*a*) designatæ, indices vero (*n*) æquales per *b* designatæ, *r* & *s* indices æquales per *c* & *d* respective designatæ, &c. tum summa prædicta per contentum $1.2.3...m \times 1.2.3...n \times 1.2.3...r \times 1 \times 2 \times 3...s \times \&c.$ dividenda est.

Ex. Sit æquatio $x^7 - 6x^6 + 9x^5 + 10x^4 - 45x^3 + 54x^2 - 29x + 16 = 0$, cujus radices sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$; invenire summam $\alpha^2\beta^3 + \beta^2\alpha^3 + \alpha^2\gamma^3 + \gamma^2\alpha^3 + \beta^2\gamma^3 + \gamma^2\beta^3 + \alpha^2\delta^3 + \delta^2\alpha^3 + \beta^2\delta^3 + \delta^2\beta^3 + \gamma^2\delta^3 + \delta^2\gamma^3 + \&c.$

per problema primum inveniatur

$$S_2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2 + \&c. = p^2 - 2q = 18$$

$$S_3 = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \epsilon^3 + \&c. = p^3 - 3pq + 3r = 24$$

$$S_2 + 3 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2 + \&c. = p^2 - 5p^3q + 5p^2r - 5ps + 5t = 216. \\ + 5pq^2 - 5qr$$

& in hoc casu erit

$$A = S_2 \times S_3 = 18 \times 24 = 432. \quad B = S_2 + 3 = 216$$

& consequenter per problema summa quæsita erit $A - B = 432 - 216 = 216.$

Ex. 2.

Ex. 2. Sit æquatio $x^6 - 5x^5 + 4x^4 + 10x^3 + 11x^2 - 5x + 6 = 0$,
cujus radices sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$

invenire summam quantitatum

$$\alpha^2\beta^3\gamma^4 + \alpha^2\gamma^3\beta^4 + \beta^2\alpha^3\gamma^4 + \beta^2\gamma^3\alpha^4 + \gamma^2\alpha^3\beta^4 + \gamma^2\beta^3\alpha^4 +$$

$$\alpha^2\beta^3\delta^4 + \delta^2\alpha^3\beta^4 + \delta^2\beta^3\alpha^4 + \&c.$$

per problema primum inveniantur

$$S_2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2 + \&c. = 17$$

$$S_3 = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \epsilon^3 + \&c. = 35$$

$$S_4 = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \epsilon^4 + \&c. = 101$$

&

$$S_{2+3} = \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \epsilon^5 + \&c. = 275$$

$$S_{2+4} = \alpha^6 + \beta^6 + \gamma^6 + \delta^6 + \epsilon^6 + \&c. = 797$$

$$S_{3+4} = \alpha^7 + \beta^7 + \gamma^7 + \delta^7 + \epsilon^7 + \&c. = 2315$$

&

$$S_{2+3+4} = \alpha^9 + \beta^9 + \gamma^9 + \delta^9 + \epsilon^9 + \&c. = 20195:$$

et consequenter fient

$$A = S_2 \times S_3 \times S_4 = 17 \times 35 \times 101 = 60095$$

$$B = S_{2+3} \times S_4 + S_{2+4} \times S_3 + S_{3+4} \times S_2 = 275 \times 101 + 797 \times 35 + 2315 \times 17 = 95025$$

$$C = S_{2+3+4} = 20195$$

et per problema summa quæsitæ erit $A - B + 2C = 5460$.

Haud raro multo facilius e coefficientibus datæ æquationis ad quæsitum aggregatum progredi licet.

Hujusce generis solutionem subsequens semper præbebit methodus.

Sit index a minor quam b , b quam c , c quam d , d quam e , $\&c.$ & sit numerus indicum $a, b, c, d, e, \&c. = v$, & data æquatio

$$x^n - px^{n-1} + qx^{n-2} - \dots \pm Px^{n-v+3} \mp Qx^{n-v+2} \pm Rx^{n-v+1} \mp Sx^{n-v} \pm \&c. = 0.$$

Et primus terminus solutionis erit

$$S^a \times R^{b-a} \times Q^{c-b} \times P^{d-c} \times \&c. \text{ \& eodem modo dilui possunt, quæcun-}$$

que volueris, resultantia contenta. E. G.

Sit æquatio $x^4 - px^3 + qx^2 - rx + s = 0$, cujus radices sint $\alpha, \beta, \gamma, \delta$:
invenire aggregatum e singulis contentis $\alpha\beta^2\gamma^3 + \alpha\gamma^2\beta^3 + \beta\alpha^2\gamma^3 +$
 $\beta\gamma^2\alpha^3 + \gamma\alpha^2\beta^3 + \gamma\beta^2\alpha^3 + \alpha\beta^2\delta^3 + \alpha\delta^2\beta^3 + \beta\alpha^2\delta^3 + \&c.$

Hic indices a, b, c respective sunt 1, 2, 3; & numerus indicum $v=3$; & exinde $n-v=4-3=1$; & $S=r, R=q, Q=p$; & $S^a \times R^b \times Q^c = rqp$; & resultantia contenta e multiplicatione factorum $r \times q \times p =$
 $\frac{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta \times \alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \gamma\delta \times}{\alpha + \beta + \gamma + \delta}$ erunt $\alpha\beta^2\gamma^3 + \alpha\gamma^2\beta^3 + \beta\alpha^2\gamma^3 + \&c. + 3 \times$
 $\frac{\alpha^2\beta^2\gamma^2 + \alpha^2\beta^2\delta^2 + \alpha^2\gamma^2\delta^2 + \beta^2\gamma^2\delta^2 + 8\alpha\beta\gamma\delta \times \alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \gamma\delta}{(8sq) + 3\alpha\beta\gamma\delta \times \alpha^2 + \beta^2 + \gamma^2 + \delta^2} (3sp^2 - 6sq)$; diluere autem contentum
 $3 \times \alpha^2\beta^2\gamma^2 + \alpha^2\beta^2\delta^2 + \alpha^2\gamma^2\delta^2 + \beta^2\gamma^2\delta^2$, cujus indices a, b, c sunt
 respective 2, 2, 2, & eorum numerus $v=3$; unde $n-v=4-3=1$;
 & $S=r, \& S^a \times R^b \times Q^c = S^2 R^0 Q^0 = S^2 = r^2 = \frac{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta}{\alpha + \beta + \gamma + \delta} =$
 $\frac{\alpha^2\beta^2\gamma^2 + \alpha^2\beta^2\delta^2 + \alpha^2\gamma^2\delta^2 + \beta^2\gamma^2\delta^2 + 2\alpha\beta\gamma\delta \times \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta}{(2sq)}$;
 & exinde $\alpha^2\beta^2\gamma^2 + \alpha^2\beta^2\delta^2 + \alpha^2\gamma^2\delta^2 + \beta^2\gamma^2\delta^2 = r^2 - 2qs$; & summa
 quæfita $\alpha\beta^2\gamma^3 + \alpha\gamma^2\beta^3 + \beta\alpha^2\gamma^3 + \beta\gamma^2\alpha^3 + \&c. = rqp - 8sq -$
 $3sp^2 + 6sq - 3 \times r^2 - 2qs = rqp + 4qs - 3r^2 - 3sp^2$.

Ex. 2. Datâ æquatione $x^n - px^{n-1} + qx^{n-2} - \&c. \dots O x^{n-r+1} \pm$
 $P x^{n-r+4} \mp Q x^{n-r+3} \pm R x^{n-r+2} \mp S x^{n-r+1} \pm T x^{n-r} \mp \&c. \dots \pm$
 $ax^{n-m-r} \mp bx^{n-m-r-1} \pm cx^{n-m-r-2} \mp dx^{n-m-r-3} \pm ex^{n-m-r-4} \mp fx^{n-m-r-5} \pm$
 $\&c. = 0$, cujus radices sint $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \&c.$ invenire summam
 e singulis quantitatibus hujusce generis

$\alpha^2\beta^2\gamma^2\delta^2 \&c. \times \epsilon\zeta\eta \&c. + \alpha^2\beta^2\gamma^2\epsilon^2 \&c. \times \zeta\eta\theta \&c. + \alpha^2\beta^2\gamma^2\zeta^2 \&c. \times$
 $\delta\eta\theta \&c. + \beta^2\gamma^2\delta^2\epsilon^2 \&c. \times \zeta\eta\theta \&c. + \&c.$

in quibus singulis quantitatibus numerus literarum, quæ duas ha-
 beant dimensiones, fit (r); numerus autem literarum, quæ unam so-
 lummodo habeant dimensionem, fit (m).

Multiplicetur coefficientis (T) termini (x^{n-r}) datæ æquationis =
 $\alpha\beta\gamma\delta\epsilon \&c. + \alpha\beta\gamma\delta\zeta \&c. + \alpha\gamma\delta\zeta\eta \&c. + \beta\gamma\delta\eta\theta \&c. + \&c.$, in
 quibus singulis contentis numerus literarum erit r , in coefficientem
 (a) termini (x^{n-m-r}) datæ æquationis = $\alpha\beta\gamma\delta\epsilon\zeta \&c. + \alpha\beta\gamma\delta\epsilon\theta \&c. +$
 $\alpha\beta\gamma\epsilon\zeta\theta \&c. + \beta\gamma\delta\epsilon\zeta\theta \&c. + \&c.$ in quibus singulis contentis nume-
 rus literarum erit $m+r$; & rectangulum erit

$\alpha^2\beta^2\gamma^2\delta^2 \&c. \times \epsilon\zeta\eta \&c. + \alpha^2\beta^2\gamma^2\epsilon^2 \&c. \times \zeta\eta\theta \&c. + \alpha^2\beta^2\gamma^2\zeta^2 \&c.$
 $\times \delta\eta\theta \&c. + \beta^2\gamma^2\delta^2\epsilon^2 \&c. \times \zeta\eta\theta \&c. + \&c.$ (summa quæfita)

+ m

$$+ \overline{m+2} \times A + \overline{m+3} \times \frac{m+4}{2} B + \overline{m+4} \times \frac{m+5}{2} \times \frac{m+6}{3} C + \overline{m+5} \times \frac{m+6}{2} \times \frac{m+7}{3} \times \frac{m+8}{4} D + \overline{m+6} \times \frac{m+7}{2} \times \frac{m+8}{3} \times \frac{m+9}{4} \times \frac{m+10}{5} E + \&c. \text{ ad } r \text{ terminos } (A, B, C, D, E, \&c.), \text{ ubi}$$

$A = \alpha^2 \beta^2 \gamma^2 \&c. \text{ (ad } r-1 \text{ terminos)} \times \delta \epsilon \zeta \&c. \text{ (ad } m+2 \text{ terminos)}$
 $+ \alpha^2 \beta^2 \delta^2 \&c. \text{ (ad } r-1 \text{ terminos)} \times \gamma \epsilon \zeta \&c. \text{ (ad } m+2 \text{ terminos)}$
 $+ \beta^2 \gamma^2 \delta^2 \&c. \times \zeta \eta \theta \&c. + \&c.$

$B = \alpha^2 \beta^2 \&c. \text{ (ad } r-2 \text{ terminos)} \times \gamma \delta \epsilon \zeta \eta \&c. \text{ (ad } m+4 \text{ terminos)}$
 $+ \alpha^2 \gamma^2 \&c. \text{ (ad } r-2 \text{ terminos)} \times \beta \delta \epsilon \zeta \eta \&c. \text{ (ad } m+4 \text{ terminos)}$
 $+ \beta^2 \gamma^2 \&c. \times \alpha \delta \zeta \eta \theta \&c. + \&c.$

$C = \alpha^2 \beta^2 \&c. \text{ (ad } r-3 \text{ terminos)} \times \gamma \delta \epsilon \zeta \&c. \text{ (ad } m+6 \text{ terminos)}$
 $+ \alpha^2 \gamma^2 \&c. \text{ (ad } r-3 \text{ terminos)} \times \beta \delta \epsilon \zeta \&c. \text{ (ad } m+6 \text{ terminos)} + \&c.$

$D = \alpha^2 \beta^2 \&c. \text{ (ad } r-4 \text{ terminos)} \times \gamma \delta \epsilon \zeta \eta \&c. \text{ (ad } m+8 \text{ terminos)}$
 $+ \alpha^2 \gamma^2 \&c. \text{ (ad } r-4 \text{ terminos)} \times \beta \delta \epsilon \zeta \eta \&c. \text{ (ad } m+8 \text{ terminos)}$
 $+ \&c.$

&c.

&c.

Diluantur termini $A, B, C, D, \&c.$, & resultabit summa quæsitæ

$$Ta - \overline{m+2} Sb + \overline{m+1} \times \frac{m+4}{2} Rc - \overline{m+1} \times \frac{m+2}{2} \times \frac{m+6}{3} Qd + \overline{m+1} \times \frac{m+2}{2} \times \frac{m+3}{3} \times \frac{m+8}{4} Pe - \overline{m+1} \times \frac{m+2}{2} \times \frac{m+3}{3} \times \frac{m+4}{4} \times \frac{m+10}{5} Of + \&c. \text{ ad } r+1 \text{ terminos.}$$

Generaliter termini coefficientis, cujus distantia a primo fit s , erit $\frac{m+2s}{s} \times \overline{m+1} \times \frac{m+2}{2} \times \frac{m+3}{3} \times \frac{m+4}{4} \dots \dots \frac{m+s-1}{s-1}$.

Cor. Sit a summa radicum, b summa quadratorum e singulis radicibus, c summa cuborum, d summa quadrato-quadratorum, e summa quadrato-cuborum, f summa cubo-cuborum, &c.

Summa contentorum sub singulis (m) radicibus erit æqualis fractioni, cujus numerator est

$$a^m - m \times \frac{m-1}{2} a^{m-2} b + m \times \overline{m-1} \times \frac{m-2}{3} a^{m-3} c - m \times \overline{m-1} \times \overline{m-2} \times \frac{m-3}{4} a^{m-4} d + \&c. \\ + m \times \overline{m-1} \times \frac{m-2}{2} \times \frac{m-3}{2} a^{m-4} b^2 - \dots$$

denomi-

denominator vero $1 \cdot 2 \cdot 3 \cdot 4 \dots m$.

Scribantur pro literis a, b, c, d, e , &c. literæ p, q, r, s, t , &c. respective; & hujus seriei literalia contenta eadem fient ac literalia contenta seriei in problemate primo expressæ.

Coefficiens literalis contenti ($a^{m-1} b^{m-2} c^{m-3} d^{m-4} \dots$) erit æqualis fractioni, cujus numerator est $m \times m-1 \times m-2 \times m-3 \times m-4 \times \dots$ in quo tot factores erunt, quot unitates in summa $2\pi + 3\rho + 4\sigma + \dots$; denominator vero æqualis est facto orto e continua multiplicatione contenti $2^\pi 3^\rho 4^\sigma 5^\tau \dots$ in series $1 \times 2 \times 3 \times \dots$, $1 \times 2 \times 3 \times \dots$, $1 \times 2 \times 3 \times \dots$, &c. quarum prima continet tot terminos, quot sunt unitates in π ; secunda, quot sunt unitates in ρ ; tertia, quot sunt in σ ; & sic deinceps.

PROB. IV.

Datâ æquatione, invenire summam quarumcunque algebraicarum functionum e radicibus compositarum.

1. Si rationales & integræ sint functiones: methodis superius traditis derivari potest summa e singulis earum valoribus confecta.

2. Si functiones sint rationaliter fractionales; ad communem denominatorem reducendæ sunt fractiones; & per methodos prædictas, tam numerator, quam denominator investigari possit. E. G.

Sit æquatio $x^3 - 7x + 6 = 0$, sintque ejus radices α, β, γ ; & fractionalis functio $\frac{3x^2 + 5}{4x + 1}$.

Invenire summam fractionum $\frac{3\alpha^2 + 5}{4\alpha + 1}, \frac{3\beta^2 + 5}{4\beta + 1}, \frac{3\gamma^2 + 5}{4\gamma + 1}$.

Reducantur hæ fractiones ad communem denominatorem, & fiunt

$$\frac{48 \times \alpha\beta\gamma \times \alpha + \beta + \gamma + 12 \times \alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta}{64 \times \alpha\beta\gamma + 3 \times \alpha^2 + \beta^2 + \gamma^2 + 80 \times \alpha\beta + \alpha\gamma + \beta\gamma + 40\alpha + \beta + \gamma + 15} \\ \frac{16 \times \alpha\beta + \alpha\gamma + \beta\gamma + 4 \times \alpha + \beta + \gamma + 1}{}$$

Pro p, q, r , scribantur $0, -7$ & -6 respective, & erit denominator $64 \times \alpha\beta\gamma (64r = 64 \times -6) + 16\alpha\beta + \alpha\gamma + \beta\gamma (16 \times q = 16 \times -7) + 4 \times \alpha + \beta + \gamma (4p = 0) + 1 = -495$

et

et numerator $48 \times \alpha\beta\gamma \times \alpha + \beta + \gamma (48rp = 0) + 12 \times$
 $\alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta (12pq - 3r = 12 \times 3 \times 6)$
 $+ 3 \times \alpha^2 + \beta^2 + \gamma^2 (3 \times p^2 - 2q = 3 \times 2 \times 7) + 80 \times \alpha\beta + \alpha\gamma + \beta\gamma$
 $(80q = 80 \times -7) + 40 \times \alpha + \beta + \gamma (40 \times p = 0) + 15 = -287.$

Et consequenter summa quæsitæ est $\frac{287}{495}.$

Ex. 2. Sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ radices datæ æquationis
 $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$; & sit fractio
 $\frac{a - bx + cx^2 - dx^3 + \&c.}{A - Bx + Cx^2 - Dx^3 + \&c.}$, in quâ sint coefficientes ($a, b, c, \&c.$
 $A, B, C, \&c.$) datæ.

Invenire summam fractionum

$$\frac{a - b\alpha + c\alpha^2 - d\alpha^3 + \&c.}{A - B\alpha + C\alpha^2 - D\alpha^3 + \&c.}, \frac{a - b\beta + c\beta^2 - d\beta^3 + \&c.}{A - B\beta + C\beta^2 - D\beta^3 + \&c.}, \frac{a - b\gamma + c\gamma^2 - d\gamma^3 + \&c.}{A - B\gamma + C\gamma^2 - D\gamma^3 + \&c.},$$

Reducantur hæ fractiones ad communem denominatorem, & problematis primi & tertii ope inveniri potest & numerator & denominator.

3. Si functiones sint irrationales: tum summa e singulis valoribus quantitatum irrationalium deducta nihilo erit æqualis, quod si sint fractionales ad communem denominatorem 1^{mo} sunt reducendæ. E. G.

Sit æquatio $x^7 - 4x^6 + 3x^5 + 10x^4 - 25x^3 + 24x^2 - 11x + 12 = 0$; cujus radices sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$; invenire summam e singulis valoribus irrationalis functionis radicem

$$\alpha + \sqrt{\alpha^2 + \beta^2}, \beta + \sqrt{\alpha^2 + \beta^2}, \alpha + \sqrt{\alpha^2 + \gamma^2}, \gamma + \sqrt{\alpha^2 + \gamma^2}, \&c.$$

$$\alpha - \sqrt{\alpha^2 + \beta^2}, \beta - \sqrt{\alpha^2 + \beta^2}, \alpha - \sqrt{\alpha^2 + \gamma^2}, \gamma - \sqrt{\alpha^2 + \gamma^2},$$

summa irrationalium quantitatum nihilo semper est æqualis; summa vero rationalium erit $2 \times 6 \times \alpha + \beta + \gamma + \delta + \epsilon + \&c. = 12 \times 4 = 48.$

Ex. 2. Sit æquatio $x^3 - px^2 + qx - r = 0$, cujus radices sint α, β, γ ; invenire summam irrationalium fractionum

$$\frac{\alpha}{\alpha + \sqrt{\alpha^2 + 2\alpha}}, \frac{\alpha}{\alpha - \sqrt{\alpha^2 + 2\alpha}},$$

$$\frac{\beta}{\beta + \sqrt{\beta^2 + 2\beta}}, \frac{\beta}{\beta - \sqrt{\beta^2 + 2\beta}}, \frac{\gamma}{\gamma + \sqrt{\gamma^2 + 2\gamma}}, \frac{\gamma}{\gamma - \sqrt{\gamma^2 + 2\gamma}}.$$

Reducantur.

Reducantur hæ fractiones ad communem denominatorem, & fient

$$\frac{4\beta\gamma\alpha\alpha - 4\beta\gamma\alpha\sqrt{\alpha^2 + 2\alpha}}{-8\beta\gamma\alpha} \left(\frac{\alpha - \sqrt{\alpha^2 + 2\alpha}}{-2} \right), \frac{4\alpha\gamma\beta\beta - 4\alpha\gamma\beta\sqrt{\beta^2 + 2\beta}}{-8\alpha\beta\gamma} \left(\frac{\beta - \sqrt{\beta^2 + 2\beta}}{-2} \right)$$

$$\frac{4\alpha\gamma\alpha\alpha + 4\beta\gamma\alpha\sqrt{\alpha^2 + 2\alpha}}{-8\beta\gamma\alpha} \left(\frac{\alpha + \sqrt{\alpha^2 + 2\alpha}}{-2} \right), \frac{4\alpha\gamma\beta\beta + 4\alpha\gamma\beta\sqrt{\beta^2 + 2\beta}}{-8\alpha\beta\gamma} \left(\frac{\beta + \sqrt{\beta^2 + 2\beta}}{-2} \right)$$

$$\frac{4\alpha\beta\gamma\gamma - 4\alpha\beta\gamma\sqrt{\gamma^2 + 2\gamma}}{-8\alpha\beta\gamma} \left(\frac{\gamma - \sqrt{\gamma^2 + 2\gamma}}{-2} \right), \frac{4\alpha\beta\gamma\gamma + 4\alpha\beta\gamma\sqrt{\gamma^2 + 2\gamma}}{-8\alpha\beta\gamma} \left(\frac{\gamma + \sqrt{\gamma^2 + 2\gamma}}{-2} \right).$$

Et consequenter summa

quæsitæ erit $-\alpha + \beta + \gamma = -p$.

Cor. Hinc investigari possit aggregatum e singulis valoribus infinitæ potestatum seriei datæ æquationis radicum; si modo illa infinita potestatum series per algebraicam radicum functionem exprimi possit.

Inveniatur algebraica radicum functio, quæ exprimit infinitam potestatum seriem; & per hocce problema inveniri possit aggregatum e singulis ejus valoribus, & consequenter aggregatum e singulis valoribus infinitæ potestatum seriei.

E. G. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + \&c. = 0$. cujus radices sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ & sit a summa ejus radicum, b summa quadratorum e singulis ejus radicibus, c summa cuborum, d summa quadrato-quadratorum, e summa quadrato-cuborum, & sic in infinitum.

Invenire summam $a + b + c + d + e + \&c.$ in infinitum. Hæc infinita series exprimi possit per fractionum summam

$\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} + \frac{1}{1-\delta} + \frac{1}{1-\epsilon} + \&c. = n$; summa autem e singulis hisce fractionibus

$$\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} + \frac{1}{1-\delta} + \frac{1}{1-\epsilon} + \&c. = \frac{n - n - 1p + n - 2q - n - 3r + n - 4s - n - 5t + \&c.}{1 - p + q - r + s - t + \&c.}$$

& consequenter summa quæsitæ

$$\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} + \frac{1}{1-\delta} + \frac{1}{1-\epsilon} + \&c. = n = \frac{p - 2q + 3r - 4s + 5t - \&c.}{1 - p + q - r + s - t + \&c.}$$

Ex. 2. Invenire summam $a + 2b + 3c + 4d + 5e + \&c.$ in infinitum. Hæc

Hæc series exprimi possit per fractiones

$$\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} + \frac{1}{1-\delta} + \&c. = \frac{1}{1-\alpha} \frac{1}{1-\beta} \frac{1}{1-\gamma} \frac{1}{1-\delta} \&c.$$

quarum summa erit

$$\frac{n - n - 1p + n - 2q - n - 3r + n - 4s - n - 5t + \&c.}{1 - p + q - r + s - t \&c.} = \frac{n \times n - 1 - n - 1 \times n - 2p + n - 2 \times n - 3q - n - 3 \times n - 4r + n - 4 \times n - 5s - \&c.}{1 - p + q - r + s - t + \&c.} = \frac{n - n - 1p + n - 2q - n - 3r + \&c.}{1 - p + q - r + \&c.}$$

PROB. V.

Datâ unâ, duabus, vel pluribus æquationibus; invenire æquationem, cujus radices sint quæcunque algebraica radicum datarum æquationum functio.

Scribantur $\alpha, \beta, \gamma, \delta, \&c.$ pro radicibus unius datæ æquationis; $A, B, \Gamma, \Delta, \&c.$ pro radicibus alterius datæ æquationis; & sic de reliquis.

Et ex datâ algebraicâ relatione inter radices datæ æquationis, vel datarum æquationum, & radices quæsitæ æquationis, inveniantur omnes quæsitæ æquationis radices; & e problematibus præcedentibus inveniatur e singulis hisce radicibus aggregatum, quod est coefficientis secundi quæsitæ æquationis termini.

Ducantur quæque duæ radices quæsitæ æquationis in se invicem; & e prædictis problematibus inveniatur aggregatum rectorum resultantium; & erit coefficientis tertii quæsitæ æquationis termini.

Ducantur quæque tres, quatuor, quinque, &c. radices quæsitæ æquationis continuo in se invicem; & ex iisdem problematibus inveniantur aggregata resultantium contentorum; & hæc aggregata erunt respective coefficientes quarti, quinti, sexti, &c. quæsitæ æquationis terminorum.

Hinc dilucide constat methodus inveniendi quæsitæ æquationis coefficientes, & consequenter ipsam æquationem.

Hujusce generis problemata sæpe resolvi possunt e reductione duarum, vel plurium æquationum in unam, ita ut incognitæ quantitates exterminentur.

Ex. 1. Sint $\alpha, \beta, \gamma, \delta, \&c.$ radices datæ æquationis

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0,$$

 invenire æquationem

$$v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + Sv^{n-4} - \&c. = 0,$$
 cujus radices
 sint $(\alpha^m, \beta^m, \gamma^m, \delta^m, \&c. \text{ respective})$ m potestates datæ æquationis
 radicum.

Scribatur

$$a = \alpha^m + \beta^m + \gamma^m + \delta^m + \&c.$$

$$b = \alpha^{2m} + \beta^{2m} + \gamma^{2m} + \delta^{2m} + \&c.$$

$$c = \alpha^{3m} + \beta^{3m} + \gamma^{3m} + \delta^{3m} + \&c.$$

$$d = \alpha^{4m} + \beta^{4m} + \gamma^{4m} + \delta^{4m} + \&c.$$

$$\&c. \quad \&c.$$

E problematis tertii corollario inveniantur quæsitæ æquationis
 coefficientes ($P, Q, R, S, \&c.$), & erunt respective

$$P = \alpha^m + \beta^m + \gamma^m + \delta^m + \&c. = a$$

$$Q = \alpha^m \times \beta^m + \alpha^m \times \gamma^m + \beta^m \times \gamma^m + \alpha^m \times \delta^m + \beta^m \times \delta^m + \gamma^m \times \delta^m + \&c. = \frac{a^2 - b}{2}$$

$$R = \alpha^m \times \beta^m \times \gamma^m + \alpha^m \times \beta^m \times \delta^m + \alpha^m \times \gamma^m \times \delta^m + \beta^m \times \gamma^m \times \delta^m + \&c. = \frac{a^3 - 3ab + 2c}{6}$$

$$S = \alpha^m \times \beta^m \times \gamma^m \times \delta^m + \&c. = \frac{a^4 - 6a^2b + 8ac - 6d + 3b^2}{24}$$

& sic de reliquis coefficientibus.

E problemate tertio deduci potest alia hujusce exempli solutio.

Sint $b, k, l, o, \&c.$ omnes diversæ radices (m) potestatis unitatis:

E. G. si (m) sit 3; b, k, l erunt 1, $\frac{-1 + \sqrt{-3}}{2}$, $\frac{-1 - \sqrt{-3}}{2}$ respective.

Ex hypothesi $x^m = v$, & consequenter $b v^{\frac{1}{m}} = x$, $k v^{\frac{1}{m}} = x$, $l v^{\frac{1}{m}} = x$,
 $o v^{\frac{1}{m}} = x$, &c.

Quibus valoribus in datâ æquatione pro quantitate (x) substitutis,
 resultant æquationes

$$b^n v^{\frac{n}{m}} - p \times b^{n-1} v^{\frac{n-1}{m}} + q \times b^{n-2} v^{\frac{n-2}{m}} - r \times b^{n-3} v^{\frac{n-3}{m}} + \&c. = 0$$

$$k^n v^{\frac{n}{m}} - p \times k^{n-1} v^{\frac{n-1}{m}} + q \times k^{n-2} v^{\frac{n-2}{m}} - r \times k^{n-3} v^{\frac{n-3}{m}} + \&c. = 0$$

$$l^n v^{\frac{n}{m}} - p \times l^{n-1} v^{\frac{n-1}{m}} + q \times l^{n-2} v^{\frac{n-2}{m}} - r \times l^{n-3} v^{\frac{n-3}{m}} + \&c. = 0$$

$$o^n v^{\frac{n}{m}} - p \times o^{n-1} v^{\frac{n-1}{m}} + q \times o^{n-2} v^{\frac{n-2}{m}} - r \times o^{n-3} v^{\frac{n-3}{m}} + \&c. = 0$$

& sic deinceps.

Ducantur hæ æquationes in se invicem; & æquatio resultans erit $v^n - P v^{n-1} + Q v^{n-2} - R v^{n-3} + S v^{n-4} - \&c. = 0$, cujus radices sunt (m) potestates datæ æquationis radicum.

E problemate tertio investigari possunt coefficientes $(P, Q, R, S, \&c.)$; & quo minor est numerus (m) , eo magis simplex est lex, quam observant quæsitæ æquationis coefficientes. E. G. Sit $m = 2$, & coefficientes erunt 1, vel 2; sit $m = 3$, & coefficientes erunt 1, vel 3; sit $m = 4$, & coefficientes erunt 1, vel 2, vel 4; sit $m = 5$, & coefficientes erunt 1, vel 5; &c.

E problemate primo alia confectatur solutio.

Cor. Hinc deleri possunt quæcunque surdæ quantitates e datâ æquatione.

Sit surda quantitas delenda $x^{\frac{1}{n}}$, & data æquatio $Ax^{\frac{n-1}{n}} - Bx^{\frac{n-2}{n}} + Cx^{\frac{n-3}{n}} - Dx^{\frac{n-4}{n}} + \&c. = Q$; assumatur $v = x^{\frac{1}{n}}$, & inveniatur ab exemplo æquatio, cujus radix est v^n ; pro v^n substituatur x , & quod oportet, factum est.

Ex. 2. Invenire æquationem, cujus radices sint $(\alpha + \beta, \alpha + \gamma, \beta + \gamma, \alpha + \delta, \beta + \delta, \gamma + \delta, \&c.)$ summæ e quibuscunque duabus datæ æquationis radicibus: numerus quæsitæ æquationis radicum, & consequenter ejus dimensiones, erit æqualis fractioni $n \times \frac{n-1}{2}$.

Summa radicum est $\frac{n-1}{2} \times \alpha + \beta + \gamma + \delta + \epsilon + \&c. = \frac{n-1}{2} \times p$; quæ est coefficientis secundi quæsitæ æquationis termini.

Ducantur quæque duæ radices quæsitæ æquationis in se invicem, & aggregatum resultantium rectangulorum $(\alpha + \beta \times \alpha + \gamma, \alpha + \beta \times \alpha + \delta, \alpha + \gamma \times \alpha + \delta, \alpha + \beta \times \gamma + \delta, \&c.)$ erit

$$\frac{n-1}{2} \times \frac{n-2}{2} \times \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \&c. + n \times \frac{n-2}{2} \alpha \beta + \alpha \gamma + \beta \gamma + \alpha \delta + \beta \delta + \gamma \delta + \&c.$$

e problemate primo prodit

$$\frac{n-1}{1} \times \frac{n-2}{2} \times \overline{a^2 + \beta^2 + \gamma^2 + \delta^2 + \&c.} = \frac{n-1}{1} \times \frac{n-2}{2} \times \overline{p^2 - 2q}$$

$$\frac{n \times n-2 \times \overline{a\beta + a\gamma + \beta\gamma + a\delta + \beta\delta + \gamma\delta + \&c.}}{1} = \frac{n \times n-2 \times q}{1}$$

quare aggregatum erit

$$\frac{n-1}{1} \times \frac{n-2}{2} \overline{p^2 + n-2 \times q}$$

Hoc aggregatum est coefficientis tertii quæsitæ æquationis termini.

Ducantur quæque tres radices quæsitæ æquationis in sese

($\overline{a+\beta} \times \overline{a+\gamma} \times \overline{a+\delta}$, $\overline{a+\beta} \times \overline{a+\gamma} \times \overline{\beta+\delta}$, $\overline{a+\beta} \times \overline{a+\gamma} \times \overline{\gamma+\delta}$, $\overline{a+\beta} \times \overline{\beta+\gamma} \times \overline{a+\delta}$, &c.)
et aggregatum horum contentorum est

$$\frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \overline{a^3 + \beta^3 + \gamma^3 + \delta^3 + \&c.} + \frac{n-2}{2} \times \frac{n^2-2n-1}{2} \times \overline{a^2\beta + a^2\gamma + a^2\delta + \beta^2a + \beta^2\gamma + \beta^2\delta}$$

$$+ \overline{\gamma^2a + \gamma^2\beta + \&c.} + \frac{n-1}{1} \times \frac{n^2-2n-2}{2} \times \overline{a\beta\gamma + a\beta\delta + a\gamma\delta + \beta\gamma\delta + \&c.}$$

e lemme primo & secundo colligitur

$$\frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \overline{a^3 + \beta^3 + \gamma^3 + \delta^3 + \&c.} = \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \overline{p^3 - 3pq + 3r}$$

$$\frac{n-2}{2} \times \frac{n^2-2n-1}{2} \times \overline{a^2\beta + a^2\gamma + \beta^2\gamma + a^2\delta + \beta^2\delta + \&c.} = \frac{n-2}{2} \times \frac{n^2-2n-1}{2} \times \overline{pq - 3r}$$

$$\frac{n-1}{1} \times \frac{n^2-2n-2}{2} \times \overline{a\beta\gamma + a\beta\delta + a\gamma\delta + \beta\gamma\delta + \&c.} = \frac{n-1}{1} \times \frac{n^2-2n-2}{2} \times \overline{r}$$

quare aggregatum est

$$\frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} \overline{p^3} + \frac{n-2}{2} \times \overline{pq} + \frac{n-4}{1} \times \overline{r}$$

Hoc vero aggregatum est coefficientis quarti quæsitæ æquationis termini.

Eodem modo inveniri possunt reliquæ coefficientes: fit v radix quæsitæ æquationis; & æquatio quæsitæ erit

$$\overline{v^{n \times \frac{n-1}{2}} n-1 \times p v^{n \times \frac{n-1}{2}-1} + \frac{n-1}{1} \times \frac{n-2}{2} p^2 + \frac{n-2}{2} q \times v^{n \times \frac{n-1}{2}-2}}$$

$$\frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} p^3 + \frac{n-2}{2} pq + \frac{n-4}{1} r \times v^{n \times \frac{n-1}{2}-3} + \&c. = 0.$$

Ex. 3. Data æquatione $x^n - p x^{n-1} + q x^{n-2} - r x^{n-3} + s x^{n-4} - t x^{n-5} + \&c. = 0$, cujus radices sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$; invenire æquationem, cujus radices sint $\overline{\alpha-\beta}, \overline{\alpha-\gamma}, \overline{\beta-\gamma}, \overline{\alpha-\delta}, \overline{\beta-\delta}, \overline{\gamma-\delta}, \&c.$

Summa radicum

$$\frac{n-1}{1} \times \overline{\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \&c.} (\frac{n-1}{1} p^2 - 2 \times \frac{n-1}{1} q)$$

$-2\alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \gamma\delta + \&c.$ ($-2q$) erit $n-1 p^2 - 2nq$; quæ est coefficientis secundi æquationis termini.

Ducantur quæque duæ radices quæsitæ æquationis in sese, & resultant quantitates ($\alpha^2 + \beta^2 - 2\alpha\beta \times \alpha^2 + \gamma^2 - 2\alpha\gamma$, $\alpha^2 + \delta^2 - 2\alpha\delta \times$

$\beta^2 + \delta^2 - 2\beta\delta$, $\alpha^2 + \delta^2 - 2\alpha\delta \times \alpha^2 + \gamma^2 - 2\alpha\gamma$, $\beta^2 + \gamma^2 - 2\beta\gamma \times$

$\beta^2 + \delta^2 - 2\beta\delta$, &c.) $n-1 \times \frac{n-2}{2} \times \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \&c.$

$-2 \times n - 2 \times \alpha^3 \times \beta + \gamma + \delta + \&c. + \beta^3 \times \alpha + \gamma + \delta + \&c. +$

$\gamma^3 \times \alpha + \beta + \delta + \&c. + \delta^3 \times \alpha + \beta + \gamma + \&c. + \&c. - 2 \times n - 3 \times$

$\alpha^2 \times \beta\gamma + \beta\delta + \gamma\delta + \&c. + \beta^2 \times \alpha\gamma + \alpha\delta + \gamma\delta + \&c. + \gamma^2 \times \alpha\delta + \alpha\delta + \beta\delta + \&c. +$

$\delta^2 \times \alpha\beta + \alpha\gamma + \beta\gamma + \&c. + \&c. + 12 \times \alpha\beta\gamma\delta + \&c. + n \times n - 2 \times$

$\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + \alpha^2\delta^2 + \beta^2\delta^2 + \&c.$

e problematibus præcedentibus

$n-1 \times \frac{n-2}{2} \times \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \&c. = n-1 \times \frac{n-2}{2} \times p^4 - 4p^2q + 4pr - 4s$

$+ 2q^2$

$-2 \times n - 2 \times \alpha^3 \times \beta + \gamma + \delta + \&c. + \beta^3 \times \alpha + \gamma + \delta + \&c. + \gamma^3 \times \alpha + \beta + \delta + \&c. + \&c. = -2 \times n - 2 \times p^3q - pr + 4s$

$- 2q^2$

$-2 \times n - 3 \times \alpha^2 \times \beta\gamma + \beta\delta + \gamma\delta + \&c. + \beta^2 \times \alpha\gamma + \alpha\delta + \beta\delta + \&c. + \gamma^2 \times \alpha\delta + \alpha\delta + \beta\delta + \&c. + \&c. = -2 \times n - 3 \times pr - 4s$

$+ n \times n - 2 \times \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + \alpha^2\delta^2 + \beta^2\delta^2 + \delta^2\gamma^2 + \&c. + \&c. = -n \times n - 2 \times 2rp - 2s - q^2$

$+ 12 \times \alpha\beta\gamma\delta + \alpha\beta\gamma + \&c. = 12 \times s$

Quare aggregatum est $n-1 \times \frac{n-2}{2} p^4 - 2n \times n - 2p^2q - 2 \times n - 3pr +$

$2ns + 2n + 3 \times n - 2q^2$, sed hoc aggregatum est coefficientis tertii datæ æquationis termini: sit v radix quæsitæ æquationis, & æquatio quæsi-

ta erit

$n \times \frac{n-1}{2} v^{n-1} - (n-1)p^2 - 2nq \times v + \frac{n-2}{2} p^4 - 2n \times n - 2p^2q - 2 \times$

$n - 3pr + 2ns + 2n + 3 \times n - 2q^2 \times v - \&c. = 0.$

Ex. 4.

Coefficiens literalis contenti $p^{n-m} q^{\mu} r^{\nu} s^{\pi} \&c.$ erit æqualis fractioni, cujus numerator est

$$n \times n - m + 1 \times n - m + 2 \times n - m + 3 \dots n - m + \mu + \nu + \pi + \rho + \&c. + 2 \\ \times n - 1 \times n - 2 \times n - 3 \times n - 4 \times \&c. \text{ et denominator } 1. 2. 3. \dots \mu \times \\ 1. 2. 3. \dots \nu \times 1. 2. 3. \dots \pi \times \&c. \times n^{n-m} \times n^{\mu+\nu+\pi+\rho+\&c.}$$

Si vero summa $\mu + \nu + \pi + \rho + \&c.$ sit vel 0, vel 1; in priori casu coefficiens quantitatis (p^n) erit $\frac{n-1}{n}$; in posteriori casu coefficiens

erit $\frac{n-1}{n^{n-m}}$. Coefficiens autem quantitatis Q erit $n-1$.

Ex. 5. Sit æquatio

$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$
& generaliter per hoc problema inveniri potest æquatio

$v^n + Pv^{n-1} + Qv^{n-2} + \&c. \dots Rv^2 + Sv + T = 0$
cujus radix $v = x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + \&c.$

Cor. Sit ultimus terminus (T) resultantis æquationis nihilo æqualis, & simplicem divisorem necessario habent duæ quantitates

$$x^n - px^{n-1} + qx^{n-2} - \&c. \\ \& x^n + ax^{n-1} + bx^{n-2} + \&c.;$$

sint duo ultimi resultantis æquationis termini (T & S) nihilo æquales, & saltem quadraticum divisorem habent prædictæ quantitates; sint tres ultimi termini nihilo æquales, & saltem cubicum divisorem habet resultans æquatio, & sic deinceps.

Et sic de communibus plurium quantitatum divisoribus inveniendis.

Ex. 6. Datâ æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$, cujus radices sint $\alpha, \beta, \gamma, \delta, \&c.$; invenire æquationem, cujus radices sint $\alpha - \beta, \alpha - \gamma, \beta - \gamma, \alpha - \delta, \beta - \delta, \&c.$

$\beta - \alpha, \gamma - \alpha, \gamma - \beta, \delta - \alpha, \delta - \beta, \&c.$
numerus radicum erit $n \times n - 1$.

Summa radicum erit nihilo æqualis, quæ est coefficiens secundi æquationis termini: etiamque constat coefficientes quarti, sexti, & cujuslibet paris æquationis termini nihilo æquales esse.

Ducanter

Ex. 4. Sit æquatio $x^{n-1} - \frac{n-1}{n} \times p x^{n-2} + \frac{n-2}{n} \times q x^{n-3} - \frac{n-3}{n} \times$
 $r x^{n-4} + \frac{n-4}{n} \times s x^{n-5} - \frac{n-5}{n} \times t x^{n-6} + \&c. = 0$; cujus radices
sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ invenire æquationem, cujus radices sint respective
 $\alpha^n - p \alpha^{n-1} + q \alpha^{n-2} - r \alpha^{n-3} + s \alpha^{n-4} - t \alpha^{n-5} + \&c. + \mathcal{Q},$
 $\beta^n - p \beta^{n-1} + q \beta^{n-2} - r \beta^{n-3} + s \beta^{n-4} - t \beta^{n-5} + \&c. + \mathcal{Q},$
 $\gamma^n - p \gamma^{n-1} + q \gamma^{n-2} - r \gamma^{n-3} + s \gamma^{n-4} - t \gamma^{n-5} + \&c. + \mathcal{Q},$
 $\delta^n - p \delta^{n-1} + q \delta^{n-2} - r \delta^{n-3} + s \delta^{n-4} - t \delta^{n-5} + \&c. + \mathcal{Q},$
 $\&c. \qquad \qquad \qquad \&c.$

Summa radicum erit

$$\begin{aligned} \alpha^n + \beta^n + \gamma^n + \delta^n + \&c. &= \frac{n-1}{n} p^n - n \times \frac{n-1}{n^{n-2}} \times \frac{n-2}{n} q p^{n-2} + n \times \frac{n-1}{n^{n-3}} \times \frac{n-3}{n} r p^{n-3} - n \times \frac{n-1}{n^{n-4}} \times \frac{n-4}{n} s p^{n-4} \\ &\quad + n \times \frac{n-3}{2} \times \frac{n-1}{n^{n-4}} \times \frac{n-2}{n} p^{n-2} \\ - p \times \alpha^{n-1} + \beta^{n-1} + \gamma^{n-1} + \delta^{n-1} + \&c. &= - \frac{n-1}{n^{n-1}} p^n + \frac{n-1}{n-1} \times \frac{n-1}{n^{n-3}} \times \frac{n-2}{n} q p^{n-2} - \frac{n-1}{n-1} \times \frac{n-1}{n^{n-4}} \times \frac{n-3}{n} r p^{n-3} \\ &\quad + q \times \alpha^{n-2} + \beta^{n-2} + \gamma^{n-2} + \delta^{n-2} + \&c. = + \frac{n-1}{n^{n-2}} q p^{n-2} - \frac{n-2}{n-2} \times \frac{n-1}{n^{n-4}} \times \frac{n-2}{n} p^{n-2} \\ - r \times \alpha^{n-3} + \beta^{n-3} + \gamma^{n-3} + \delta^{n-3} + \&c. &= - \frac{n-1}{n^{n-3}} p^{n-3} + \&c. \\ + s \times \alpha^{n-4} + \beta^{n-4} + \gamma^{n-4} + \delta^{n-4} + \&c. &= + \frac{n-1}{n^{n-4}} p^{n-4} s - \&c. \\ &\qquad \qquad \qquad \&c. \qquad \qquad \qquad \&c. \end{aligned}$$

Quare aggregatum erit

$$\begin{aligned} &= - \frac{n-1}{n^n} p^n + \frac{n-1}{n^{n-2}} \times q p^{n-2} - \frac{n-1}{n^{n-3}} r p^{n-3} + \frac{n-1}{n^{n-4}} s p^{n-4} - \frac{n-1}{n^{n-5}} t p^{n-5} + \&c. \\ &\quad - \frac{1}{2} n \times \frac{n-2}{n^2} \times \frac{n-2}{n^{n-4}} q^2 p^{n-4} + n \times \frac{n-2 \times n-3}{n \times n} \times \frac{n-1}{n^{n-5}} q r p^{n-5} - \&c. \end{aligned}$$

Lex, quam observat hæc series, exprimens secundi termini resultantis æquationis coefficientem, vel contenta literalia, vel coefficientes respicit.

Contenta literalia eandem, quam contenta literalia seriei in problemate primo traditæ, observant legem.

Coefficiens

$$\times \frac{n-4}{n} s p^{n-4} + \&c.$$

$$\times \frac{n-2}{n^2} \times q^2 \times p^{n-4}$$

$$\times \frac{n-3}{n} r p^{n-3} + \frac{n-1}{n} \times \frac{n-1}{n^{n-5}} \times \frac{n-4}{n} s p^{n-4} - \&c.$$

$$- \frac{n-1}{2} \times \frac{n-4}{2} \times \frac{n-1}{n^{n-5}} \times \frac{n-2}{n^2} q^2 p^{n-4}$$

$$\times \frac{n-2}{n} \times q^2 p^{n-4} + \&c.$$

. &c.

Ducantur quæque duæ radices quæsitæ æquationis in se invicem, & resultant rectangula $-n-1 \times a^2 + \beta^2 + \gamma^2 + \delta^2 + \&c. + 2 \times a\beta + a\gamma + \beta\gamma + a\delta + \beta\delta + \gamma\delta + \&c. = -n-1 p^2 + 2nq$; quæ erit coefficientis tertii æquationis termini.

Sit v radix quæsitæ æquationis, & quæsitæ æquatio erit

$$v^{n-1} + n-1 p^2 - 2nq \times v^{n-2} + n-1 \times \frac{n-2}{2} p^4 - 2n \times n-2 p^2 q - 2 \times n-3 pr + 2ns + 2n+3 \times n-2 q^2 \times v^{n-1-4} + \&c. = 0.$$

Cor. Radices hujus æquationis erunt quadraticæ radices æquationis resultantis in exemplo tertio contentæ.

Ex. 7. Sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ radices datæ æquationis

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + \&c. = 0,$$

& $A, B, \Gamma, \Delta, \&c.$ radices datæ æquationis

$$z^m - Pz^{m-1} + Qz^{m-2} - Rz^{m-3} + Sz^{m-4} - \&c. = 0$$

invenire æquationem, cujus radices sint ($\alpha A, \alpha B, \alpha \Gamma, \alpha \Delta, \&c. \beta A, \beta B, \beta \Gamma, \beta \Delta, \&c. \gamma A, \gamma B, \gamma \Gamma, \gamma \Delta, \&c.$) rectangula sub singulâ unius datæ æquationis radice & singulâ alterius radice contenta.

Numerus radicum quæsitæ æquationis, & consequenter ejus dimensiones, æquales erunt rectangulo sub indicibus ($n \times m$) datarum æquationum.

Summa radicum, & consequenter secundi termini quæsitæ æquationis coefficientis æqualis est rectangulo

$$\alpha + \beta + \gamma + \delta + \&c. \times A + B + \Gamma + \Delta + \&c. = P \times p.$$

Ducantur quæque duæ radices quæsitæ æquationis in sese ($\alpha A \times \alpha B, \alpha A \times \alpha \Gamma, \alpha A \times \alpha \Delta, \&c. \beta A \times \alpha A, \beta A \times \alpha B, \beta A \times \alpha \Gamma, \&c. \gamma A \times \alpha A, \&c.$) & rectangulorum resultantium aggregatum, seu coefficientis tertii quæsitæ æquationis termini est

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \&c. \times AB + A\Gamma + A\Delta + B\Gamma + B\Delta + \Gamma\Delta + \&c. + A^2 + B^2 + \Gamma^2 + \Delta^2 + \&c. \times \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta + \&c. + 2 \times \alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \gamma\delta + \&c. \times AB + A\Gamma + B\Gamma + A\Delta + B\Delta + \Gamma\Delta + \&c.$$

E problemate primo colligitur

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \&c. \times AB + A\Gamma + B\Gamma + A\Delta + B\Delta + \&c. = p^2 - 2q \times Q$$

$A^2 +$

$$A^2 + B^2 + \Gamma^2 + \Delta^2 + \&c. \times \alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \&c. = P^2 - 2Q \times q$$

$$2 \times \alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \&c. \times AB + AE + AD + BF + BD + \&c. = P^2 - 2Q \times q$$

Quorum aggregatum est

quæ est tertii quæsitæ æquationis termini coefficientis.

Ex eadem ratiocinandi methodo coefficientis quarti quæsitæ æquationis termini invenitur

$$P^3 r + p^3 R - 3PQr - 3pqR + pqPQ + 3Rr; \& \text{ sic deinceps.}$$

Sit v radix quæsitæ æquationis, & quæsitæ æquatio erit.

$$v^m - Ppv^{m-1} + P^2q - 2Qq \left(v^{m-1} - \frac{P^3r - 3PQr + 3Rr}{p^3R - 3pqR + pqPQ} v^{m-2} + \&c. \right) = 0.$$

SCHOLIUM.

In multis casibus subsequenti methodo inveniri possunt facilius æquationis quæsitæ coefficientes; assumantur æquationes datæ formulæ, quarum radices cognitæ sunt; & exinde inveniat quæsitæ æquatio. E pluribus hujusce generis resultantibus æquationibus inveniri possint terminorum coefficientes, & consequenter æquatio ipsa.

Hæc autem methodus plerumque magis usui inservit, cum dimensiones datæ æquationis haud generaliter assumantur.

E. G. Sit data æquatio $x^4 + qx^2 - rx + s = 0$; cujus radices sint a, β, γ, δ ; invenire æquationem, cujus radices sint $a + \beta, a + \gamma, \beta + \gamma, a + \delta, \beta + \delta, \gamma + \delta$.

Assumo æquationem biquadraticam datæ formulæ $x^4 - 6x^2 + 8x - 3 = 0$, cujus radices erunt $1, 1, 1, -3$; & radices resultantis æquationis erunt $1+1, 1+1, 1+1, 1-3, 1-3, 1-3$; vel $2, 2, 2, -2, -2, -2$; & exinde resultans æquatio erit $v^6 - 12v^4 + 48v^2 - 64 = 0$: radices $(a + \beta, a + \gamma, \&c.)$ resultantis æquationis unam solummodo habent dimensionem; & consequenter earum summa, quæ est coefficientis secundi resultantis æquationis termini, unam solummodo habebit dimensionem; sed nullus invenitur terminus in datâ æquatione, quæ non habet plures quam unam dimensionem; ergo summa radicum resultantis æquationis semper nihilo æqualis est. Summa rectangulorum sub singulis binis radicibus resultantis æqua-

æquationis, quæ est coefficientis tertii resultantis æquationis termini, duas habet dimensiones; unus autem solummodo invenitur terminus (q) in datâ æquatione, qui hæud plures quam duas habet dimensiones: ergo coefficientis tertii termini resultantis æquationis erit $a \times q$; sed coefficientis assumptæ æquationis, cujus radices cognitæ sunt, & quæ habet duas dimensiones, fuit -6 ; & coefficientis tertii termini ejus resultantis æquationis fuit -12 ; ergo $-6 (q) \times a = -12$; & consequenter $a = 2$; & quæsitæ æquatio $v^6 + 2q v^4 \&c. = 0$.

Contentum sub singulis tribus quæsitæ æquationis radicibus ($a + \beta \times a + \gamma \times \beta + \gamma + \&c.$), quod est coefficientis quarti resultantis æquationis termini, tres habet dimensiones; unus autem solummodo invenitur terminus (r) in datâ æquatione, qui tres possit conficere dimensiones; ergo coefficientis tertii resultantis æquationis termini erit $a \times r$: sed in assumptâ æquatione $r = 8$, & ejus resultantis æquationis quarti termini coefficientis fuit 0 , ergo $8 \times a = 0$, & consequenter $a = 0$.

Contentum sub singulis quatuor quæsitæ æquationis radicibus ($a + \beta \times a + \gamma \times \beta + \gamma \times a + \delta + \&c.$) quatuor habet dimensiones; e terminis datæ æquationis duobus modis confici possunt quatuor dimensiones, *i. e.* vel coefficientis (s), vel (q^2) quatuor habent dimensiones; ergo contentum sub singulis quatuor datæ æquationis radicibus, quod est coefficientis quinti resultantis æquationis termini, erit $aq^2 + bs$; & quoniam duæ sunt quantitates hujusmodi, quæ conficiunt quatuor dimensiones, necesse est, ut tot assumantur æquationes, quarum radices cognitæ sint, & exinde inveniantur earum resultantes æquationes: assumo igitur alteram æquationem $x^4 - 2x^2 + 1 = 0$; cujus radices sunt $1, 1, -1, -1$; & radices æquationis resultantis $2, 0, 0, 0, 0, -2$; & æquatio ipsa $v^6 - 4v^4 = 0$; e priori assumptâ æquatione colligitur $a \times 36 (q^2) + b \times -3 (s) = 48$, quæ est coefficientis quinti termini ejus resultantis æquationis; e posteriori vero æquatione sequitur $a \times 4 (q^2) + b \times 1 (s) = 0$; et ex his duabus æquationibus constat $a = 1$, & $b = -4$; & consequenter coefficientis quinti resul-

D

tantis

tantis æquationis termini $q^2 - 4s$: & quæſita æquatio

$$v^6 + 2qv^4 + q^2 - 4s \times v^2 \&c. = 0.$$

Contentum sub ſingulis quinque resultantis æquationis radicibus, quod eſt coeſſiciens ſexti resultantis æquationis termini, quinque habet diſenſiones; ſed quinque diſenſiones uno ſolummodo modo e termini datæ æquationis oriri poſſint, *i. e.* rectangulum $q \times r$ ſolummodo quinque habet diſenſiones, & exinde $a \times q \times r$ erit coeſſiciens ſexti æquationis termini: ergo e priori aſſumptâ æquatione $a \times - 6 \times 8$ ($a \times qr$) = 0, & exinde $a = 0$.

Contentum ſub ſex radicibus, quod eſt coeſſiciens ſeptimi resultantis æquationis termini, ſex habet diſenſiones; tot autem diſenſiones habent quantitates q^3 , r^2 , & sq , & nullæ aliæ in datâ æquatione contentæ; ergo coeſſiciens ultimi æquationis resultantis termini erit $aq^3 + br^2 + cqs$: ſed quoniam tres ſint quantitates, quæ habent ſex diſenſiones; neceſſe eſt, ut aſſumantur tres æquationes, quarum radices cognitæ ſint: aſſumo igitur tertiam æquationem $x^4 - 5x^2 + 4 = 0$, cujus radices ſunt 1, 2, -1, -2: & æquationis resultantis radices erunt 3, 0, -1, +1, 0, -3, & æquatio ipſa $v^6 - 10v^4 + 9v^2 = 0$. E priori aſſumptâ æquatione colligitur $a \times - 216 (aq^3) + b \times 64 (br^2) + c \times 18 (cqs) = - 64$ coeſſicienti ſeptimi ejus resultantis æquationis termini: e ſecundâ vero

$$a \times - 8(aq^3) + b \times 0(br^2) + c \times - 2(cqs) = 0: \text{ e tertiâ vero } a \times - 125(aq^3) + b \times 0(br^2) + c \times - 20(cqs) = 0, \text{ \& exinde } a, b, c \text{ erunt reſpective } 0, -1, 0: \text{ \& æquatio quæſita}$$

$$v^6 + 2qv^4 + q^2 - 4s \times v^2 - r^2 = 0.$$

Datâ æquatione inferioris (m) ordinis, & ex ejus resultante æquatione ſæpe facile inveniri poſſunt multi termini æquationis ſimiliter resultantis ex æquatione ſuperioris ($m+r$) ordinis.

Ducatur data æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$ in æquationem (r) diſenſionum, cujus radices cognitæ ſint; maxime frequenter præſtat, ut æquatio (r) diſenſionum ſit $x^r = 0$, cujus omnes radices ſint = 0; ex æquatione resultanti inveniatur æquatio,

cu-

cujus radices datam habeant relationem ad radices datæ æquationis ;
& ex hâc resultanti æquatione sequuntur sæpe quam plurimi termini
æquationis similiter resultantis ex æquatione $(m+r)$ dimensionum.

E. G. Sit æquatio data inferioris ordinis

$x^4 + qx^2 - rx + s = 0$, cujus

radices sint $\alpha, \beta, \gamma, \delta$; & æquatio, cujus radices sint $\alpha + \beta, \alpha + \gamma,$
 $\beta + \gamma, \alpha + \delta, \beta + \delta, \gamma + \delta$; erit

$$v^6 + 2qv^4 + q^2 - 4s \mid v^2 - r^2 = 0.$$

Ducatur data æquatio $x^4 + qx^2 - rx + s = 0$ in simplicem
æquationem ($x=0$), cujus radix fit (0); & fit æquatio $x^5 + qx^3 -$
 $rx^2 + sx = 0$, cujus radices erunt $\alpha, \beta, \gamma, \delta, 0$; & radices æquatio-
nis similiter exinde resultantis erunt $\alpha + \beta, \alpha + \gamma, \beta + \gamma, \alpha + \delta, \beta + \delta,$
 $\gamma + \delta, \alpha, \beta, \gamma, \delta$; & æquatio ipsa

$$v^4 + qv^2 - rv + s \times v^6 + 2qv^4 + q^2 - 4s \times v^2 - r^2 = 0;$$

cujus æquationis omnes termini iidem erunt, ac termini æquationis
similiter resultantis e datâ æquatione $x^5 + qx^3 - rx^2 + sx - t = 0$;
in quibus continentur solummodo literæ q, r, s .

CAP. II.

DE IMPOSSIBILIBUS RADICIBUS.

THEOR. I.

SIT æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$, cujus radix sit impossibilis quantitas $a + \sqrt{-b^2}$; & impossibilis quantitas $a - \sqrt{-b^2}$ etiam erit radix datæ æquationis.

Pro (x) in data æquatione scribantur $a + \sqrt{-b^2}$, & $a - \sqrt{-b^2}$ respective; & e priori substitutione resultat

$$\begin{aligned}
 x^n &= a^n - n \times \frac{n-1}{2} a^{n-2} b^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4} b^4 - \&c. \\
 &\quad + n a^{n-1} \sqrt{-b^2} - n \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} b^2 \sqrt{-b^2} + \&c. \\
 -px^{n-1} &= -pa^{n-1} + \overline{n-1} \times \frac{n-2}{2} pa^{n-3} b^2 - \overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \frac{n-4}{4} pa^{n-5} b^4 + \&c. \\
 &\quad - \overline{n-1} pa^{n-2} \sqrt{-b^2} + \overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} pa^{n-4} b^2 \sqrt{-b^2} - \&c. \\
 +qx^{n-2} &= +qa^{n-2} - \overline{n-2} \times \frac{n-3}{2} qa^{n-4} b^2 + \overline{n-2} \times \frac{n-3}{2} \times \frac{n-4}{3} \times \frac{n-5}{4} qa^{n-6} b^4 - \&c. \\
 &\quad + \overline{n-2} qa^{n-3} \sqrt{-b^2} - \overline{n-2} \times \frac{n-3}{2} \times \frac{n-4}{3} qa^{n-5} b^2 \sqrt{-b^2} + \&c. \\
 -rx^{n-3} &= -ra^{n-3} + \overline{n-3} \times \frac{n-4}{2} ra^{n-5} b^2 - \overline{n-3} \times \frac{n-4}{2} \times \frac{n-5}{3} \times \frac{n-6}{4} ra^{n-7} b^4 + \&c. \\
 &\quad - \overline{n-3} ra^{n-4} \sqrt{-b^2} + \overline{n-3} \times \frac{n-4}{2} \times \frac{n-5}{3} ra^{n-6} b^2 \sqrt{-b^2} - \&c. \\
 &\quad \&c. \qquad \&c.
 \end{aligned}$$

& e posteriori resultat

$$\begin{aligned}
 x^n &= a^n - n \times \frac{n-1}{2} a^{n-2} b^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4} b^4 - \&c. \\
 &\quad - n a^{n-1} \sqrt{-b^2} + n \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} b^2 \sqrt{-b^2} \\
 -px^{n-1} &= -pa^{n-1} + \overline{n-1} \times \frac{n-2}{2} pa^{n-3} b^2 - \overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \frac{n-4}{4} pa^{n-5} b^4 + \&c. \\
 &\quad + \overline{n-1} pa^{n-2} \sqrt{-b^2} - \overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} pa^{n-4} b^2 \sqrt{-b^2} \\
 +qx^{n-2} &= +qa^{n-2} - \overline{n-2} \times \frac{n-3}{2} qa^{n-4} b^2 + \overline{n-2} \times \frac{n-3}{2} \times \frac{n-4}{3} \times \frac{n-5}{4} qa^{n-6} b^4 - \&c. \\
 &\quad - \overline{n-2} qa^{n-3} \sqrt{-b^2} + \overline{n-2} \times \frac{n-3}{2} \times \frac{n-4}{3} qa^{n-5} b^2 \sqrt{-b^2} \\
 -rx^{n-3} &= -ra^{n-3} + \overline{n-3} \times \frac{n-4}{2} ra^{n-5} b^2 - \overline{n-3} \times \frac{n-4}{2} \times \frac{n-5}{3} \times \frac{n-6}{4} ra^{n-7} b^4 + \&c. \\
 &\quad + \overline{n-3} ra^{n-4} \sqrt{-b^2} - \overline{n-3} \times \frac{n-4}{2} \times \frac{n-5}{3} ra^{n-6} b^2 \sqrt{-b^2} + \&c. \\
 &\quad \&c. \qquad \&c.
 \end{aligned}$$

E priori

E priori, & posteriori substitutione eadem resultant possibiles quantitates, eademque etiam (signis vero mutatis) resultant impossibiles quantitates: & consequenter, si prior substitutio diluat & possibiles & impossibiles resultantes terminos, *i. e.* $(a + \sqrt{-b^2})$ sit radix datæ æquationis, posterior $(a - \sqrt{-b^2})$ etiam erit radix æquationis.

Cor. 1. Hinc omnis æquatio, vel habet nullas, vel parem numerum impossibilium radicum, omnis enim impossibilis radix $(a + \sqrt{-b^2})$ alteram habet impossibilem radicem $(a - \sqrt{-b^2})$ sibi ipsi cognatam.

Cor. 2. Ducantur duæ radices $x - a + \sqrt{-b^2}$, & $x - a - \sqrt{-b^2}$ in sese, & resultat quadratica rationalis quantitas $x^2 - 2ax + a^2 + b^2 = 0$; & exinde constat has radices in æquatione latere sub similitudine, vel duarum negativarum radicum, vel duarum affirmatarum, & nunquam sub similitudine unius negativæ, alterius vero affirmativæ, latere.

Cor. 3. Sit æquatio, $x^{2n} - px^{2n-1} + qx^{2n-2} - rx^{2n-3} + sx^{2n-4} - \&c. = 0$, cujus omnes radices sint impossibiles, *i. e.* sint radices $a + \sqrt{-A^2}$, $a - \sqrt{-A^2}$; $b + \sqrt{-B^2}$, $b - \sqrt{-B^2}$; $c + \sqrt{-C^2}$, $c - \sqrt{-C^2}$; $d + \sqrt{-D^2}$, $d - \sqrt{-D^2}$; $e + \sqrt{-E^2}$, $e - \sqrt{-E^2}$; &c., & consequenter ejus quadratici divisores erunt $\overline{x-a^2} + A^2$, $\overline{x-b^2} + B^2$, $\overline{x-c^2} + C^2$, $\overline{x-d^2} + D^2$, $\overline{x-e^2} + E^2$, &c. Ducantur hi divisores in sese, & resultat data æquatio,

$$\begin{aligned} & \overline{x-a^2} \times \overline{x-b^2} \times \overline{x-c^2} \times \overline{x-d^2} \times \overline{x-e^2} \times \&c. + A^2 \times \overline{x-b^2} \times \overline{x-c^2} \times \overline{x-d^2} \times \overline{x-e^2} \times \&c. + A^2 B^2 \times \overline{x-c^2} \times \overline{x-d^2} \times \overline{x-e^2} \times \&c. + \&c. \\ & + B^2 \times \overline{x-a^2} \times \overline{x-c^2} \times \overline{x-d^2} \times \overline{x-e^2} \times \&c. + A^2 C^2 \times \overline{x-b^2} \times \overline{x-d^2} \times \overline{x-e^2} \times \&c. \\ & + C^2 \times \overline{x-a^2} \times \overline{x-b^2} \times \overline{x-d^2} \times \overline{x-e^2} \times \&c. + B^2 C^2 \times \overline{x-a^2} \times \overline{x-d^2} \times \overline{x-e^2} \times \&c. \\ & + D^2 \times \overline{x-a^2} \times \overline{x-b^2} \times \overline{x-c^2} \times \overline{x-e^2} \times \&c. + A^2 D^2 \times \overline{x-b^2} \times \overline{x-c^2} \times \overline{x-e^2} \times \&c. \\ & + E^2 \times \overline{x-a^2} \times \overline{x-b^2} \times \overline{x-c^2} \times \overline{x-d^2} \times \&c. \\ & \&c. \end{aligned}$$

$$+ A^2 B^2 C^2 \times \overline{x-d^2} \times \overline{x-e^2} \times \&c. + \&c.$$

$$+ A^2 B^2 D^2 \times \overline{x-c^2} \times \overline{x-e^2} \times \&c. + \&c.$$

$$+ \&c. \quad \&c.$$

& consequenter constat data æquatio e quadratis, *i. e.* erit aggregatum

tum e diversis quadratis ($\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} \times \overline{x-e} \times \&c.$,
 $A^2 \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} \times \overline{x-e} \times \&c.$, $B^2 \times \overline{x-a} \times \overline{x-c} \times \overline{x-d} \times \overline{x-e} \times \&c.$,
 $C^2 \times \overline{x-a} \times \overline{x-b} \times \overline{x-d} \times \overline{x-e} \times \&c.$, $D^2 \times \overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-e} \times \&c.$, $\&c.$,
 $A^2 B^2 \times \overline{x-c} \times \overline{x-d} \times \overline{x-e} \times \&c. \&c.)$

quorum numerus secundum hanc deductionem erit 2^n , ni quaedam quadrata nihilo sint æqualia.

Ex. Sit æquatio $x^8 - 2x^7 + 2x^6 - 4x^5 + 28x^4 - 8x^3 + 8x^2 - 16x + 96 = 0$,
 cujus omnes radices erunt impossibiles, *i. e.* erunt $+1 + \sqrt{-1}$,
 $+1 - \sqrt{-1}$; $2 + \sqrt{-2}$, $2 - \sqrt{-2}$; $-1 + \sqrt{-1}$, $-1 - \sqrt{-1}$;
 $-1 + \sqrt{-3}$, $-1 - \sqrt{-3}$; & consequenter ejus quadratici divisores
 erunt respective $\overline{x-1} + 1$, $\overline{x-2} + 2$, $\overline{x+1} + 1$, $\overline{x+1} + 3$.

Ducantur hi divisores in sese, & resultat æquatio data e quadratis composita

$$\begin{aligned} & \overline{x-1} \times \overline{x-2} \times \overline{x+1} \times \overline{x+1} \\ & + 1 \times \overline{x-2} \times \overline{x+1} \times \overline{x+1} + 1 \times 2 \times \overline{x+1} \times \overline{x+1} + 1 \times 2 \times 1 \times \overline{x+1} \\ & + 2 \times \overline{x-1} \times \overline{x+1} \times \overline{x+1} + 1 \times 1 \times \overline{x-2} \times \overline{x+1} + 1 \times 2 \times 3 \times \overline{x+1} + 1.2.1.3 \\ & + 1 \times \overline{x-1} \times \overline{x-2} \times \overline{x+1} + 1 \times 3 \times \overline{x-2} \times \overline{x+1} + 1 \times 1 \times 3 \times \overline{x-2} \\ & + 3 \times \overline{x-1} \times \overline{x-2} \times \overline{x+1} + 2 \times 1 \times \overline{x-1} \times \overline{x+1} + 1 \times 2 \times 3 \times \overline{x-1} \\ & + 2 \times 3 \times \overline{x-1} \times \overline{x+1} \\ & + 1 \times 3 \times \overline{x-1} \times \overline{x-2} \end{aligned}$$

Cor. Æquatio, quæ habet possibilem radicem nullo modo possit
 esse quadratorum aggregatum; ni omnis radix alteram habet sibi ipsi
 æqualem.

THEOR. 2.

Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$. cujus
 radices sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ *i. e.* $\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \times \&c.$
 $= x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c.$ quarum radicum (α) ma-
 jor sit quam β , β quam γ , γ quam δ , δ quam ϵ , &c.: substituatur in
 datâ æquatione pro (x), quantitas (A) major quam maxima affirma-
 tiva radix; & omnes factores $A-\alpha$, $A-\beta$, $A-\gamma$, $A-\delta$, &c. erunt
 affirmativi; ergo eorum continuum contentum ($A-\alpha \times A-\beta \times$
 $A-\gamma \times A-\delta \times A-\epsilon \times \&c.)$ erit affirmativa quantitas.

Sit autem A limes inter α & β , *i. e.* minor quam α , major autem quam β , & consequenter $A - \alpha$ erit negativa quantitas, sed $A - \beta$, $A - \gamma$, $A - \delta$, &c. erunt affirmativæ quantitates: substituatur (A) pro incognitâ quantitate (x) in datâ æquatione, & quantitas resultans $A - \alpha \times A - \beta \times A - \gamma \times A - \delta \times \&c.$ erit negativa quantitas.

Sit A limes inter β & γ ; & $A - \alpha$, & $A - \beta$ erunt negativi factores, cæteri vero omnes $A - \gamma$, $A - \delta$, &c. affirmativi; & eorum continuum contentum erit affirmativa quantitas, & sic deinceps: & consequenter, si modo radices sint possibiles, & substituantur limites inter successivas radices pro incognitâ quantitate x , quantitates resultantes continuo mutantur de $+$ in $-$, & $-$ in $+$.

Cor. Si duæ quantitates in datâ æquatione pro incognitâ quantitate (x) substitutæ mutant quantitates resultantes de $+$ in $-$, vel $-$ in $+$; inter quantitates substitutas semper continetur impar radicum possibilibum numerus: si vero quantitates resultantes progrediantur de $+$ in $+$, vel $-$ in $-$; inter quantitates substitutas, vel nulla, vel par radicum numerus invenitur.

PROB. VI.

Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$: cujus radices successivæ sint $\alpha, \beta, \gamma, \delta, \&c.$ respectivæ; invenire æquationem, cujus radices sint limites inter successivas datæ æquationis radices α & β , β & γ , γ & δ , &c.

Ducantur termini successivi datæ æquationis in successivos arithmeticæ seriei terminos, $n, n-1, n-2, n-3, \&c.$ & resultat æquatio quæsitâ

$$nx^{n-1} - \overline{n-1} px^{n-2} + \overline{n-2} qx^{n-3} - \overline{n-3} rx^{n-4} + \&c. = 0.$$

Scribatur enim pro (x) in datâ æquatione ($z+a$), & resultat

$$\begin{aligned} x^n &= z^n + naz^{n-1} + n \times \frac{n-1}{2} a^2 z^{n-2} + \dots + n a^{n-1} z + a^n \\ -px^{n-1} &= -pz^{n-1} - \overline{n-1} \times pa z^{n-2} - \dots - \overline{n-1} pa^{n-2} z - pa^{n-1} \\ +qx^{n-2} &= + qz^{n-2} + \dots + \overline{n-2} qa^{n-3} z + qa^{n-2} \\ -rx^{n-3} &= -rz^{n-3} - \dots - \overline{n-3} ra^{n-4} z - ra^{n-3} \\ \&c. &= \&c. \end{aligned}$$

Sit

Sit 1^{mo} $-a = \alpha$ maxima radix datæ æquationis; & ultimus terminus resultantis æquationis erit $(a^n - p a^{n-1} + q a^{n-2} - r a^{n-3} + \&c. = 0)$; coefficientis autem penultimi termini $\frac{n x^{n-1} - \overline{n-1} p x^{n-2} + \overline{n-2} \times q x^{n-3} - \overline{n-3} r x^{n-4} + \&c. = \alpha - \beta \times \alpha - \gamma \times \alpha - \delta \times \alpha - \epsilon \times \&c.;$ sed omnes hæ quantitates erunt affirmativæ, ergo earum contentum erit affirmativum: si igitur substituatur in æquatione $\frac{n x^{n-1} - \overline{n-1} p x^{n-2} + \overline{n-2} \times q x^{n-3} - \overline{n-3} \times r x^{n-4} + \&c. = 0$ pro (x) maxima radix (α) datæ æquationis, resultans quantitas erit affirmativa.

Sit autem $-a = \beta$, & ultimus terminus $a^n - p a^{n-1} + q a^{n-2} - r a^{n-3} + \&c.$ erit nihilo æqualis; penultimi termini coefficientis $(\frac{n a^{n-1} - \overline{n-1} p a^{n-2} + \overline{n-2} q a^{n-3} - \overline{n-3} r a^{n-4} + \&c.)$ erit æqualis contento $\beta - \alpha \times \beta - \gamma \times \beta - \delta \times \beta - \epsilon \times \&c.:$ hujus contenti factor $\beta - \alpha$ erit negativa quantitas, omnes vero reliqui affirmativæ quantitates; ergo eorum contentum erit negativum: sit $-a = \gamma$, & $\frac{n a^{n-1} - \overline{n-1} p a^{n-2} + \overline{n-2} q a^{n-3} - \overline{n-3} r a^{n-4} + \&c. = \gamma - \alpha \times \gamma - \beta \times \gamma - \delta \times \gamma - \epsilon \times \&c.$ erit affirmativa quantitas; duo enim factores $\gamma - \alpha$ & $\gamma - \beta$ erunt negativæ quantitates, reliqui autem $(\gamma - \delta, \gamma - \epsilon, \&c.)$ affirmativæ; & consequenter eorum contentum $\gamma - \alpha \times \gamma - \beta \times \gamma - \delta \times \gamma - \epsilon \times \&c.$ erit affirmativum; & eodem modo substituuntur reliquæ radices datæ æquationis pro (x) in quantitate $\frac{n x^{n-1} - \overline{n-1} p x^{n-2} + \overline{n-2} q x^{n-3} - \overline{n-3} r x^{n-4} + \&c.;$ & quantitatuum resultantium signa continuo mutantur de $+$ in $-$, & $-$ in $+$; ergo radices datæ æquationis $x^n - p x^{n-1} + q x^{n-2} - r x^{n-3} + \&c. = 0$ erunt limites inter radices æquationis $\frac{n x^{n-1} - \overline{n-1} p x^{n-2} + \overline{n-2} q x^{n-3} - \overline{n-3} r x^{n-4} + \&c. = 0$; & vice versâ radices æquationis $\frac{n x^{n-1} - \overline{n-1} p x^{n-2} + \overline{n-2} q x^{n-3} - \overline{n-3} \times r x^{n-4} + \&c. = 0$ erunt limites inter radices æquationis $x^n - p x^{n-1} + q x^{n-2} - r x^{n-3} + \&c. = 0$.

Cor. 1. Generaliter ducantur successivi termini datæ æquationis $x^n - p x^{n-1} + q x^{n-2} - r x^{n-3} + \&c. = 0$ in successivos arithmeticae seriei $a, a + b, a + 2b, a + 3b, a + 4b, \&c.$ terminos, & resultat æquatio

quatio $ax^n - a + bpx^{n-1} + a + 2b \times qx^{n-2} - a + 3b \times rx^{n-3} + \&c. = 0$, cujus radices erunt limites inter radices datæ æquationis.

Ducatur enim data æquatio in $(a + nb)$, & resultat $\overline{a + nb}x^n - \overline{a + nb} \times px^{n-1} + \overline{a + nb} \times qx^{n-2} - \overline{a + nb} \times rx^{n-3} + \&c. = 0$, ducatur etiam æquatio prædicta $(nx^{n-1} - n-1px^{n-2} + n-2qx^{n-3} - \&c. = 0)$, cujus radices sunt limites inter radices datæ æquationis in bx , & resultat

$nbx^n - n-1bpx^{n-1} + n-2bqx^{n-2} - n-3brx^{n-3} + \&c. = 0$. quantitatum resultantium differentia erit

$ax^n - a + bpx^{n-1} + a + 2b \times qx^{n-2} - a + 3b \times rx^{n-3} + \&c. = 0$.

Successivæ autem radices ($\alpha, \beta, \gamma, \delta, \&c.$) datæ æquationis pro (x) in æquatione resultanti substitutæ, necessario producent easdem mutationes signorum de $+$ in $-$, & $-$ in $+$, quæ in æquatione

$-nbx^n + n-1bpx^{n-1} - n-2bqx^{n-2} + n-3brx^{n-3} - \&c. = 0$ observantur, & consequenter erunt limites inter ejus radices: & vice versâ radices æquationis resultantis erunt limites inter radices datæ æquationis.

Cor. 2. Æquationis resultantis $ax^n - a + bpx^{n-1} + a + 2b \times qx^{n-2} - \&c. = 0$ (n) erunt radices; ni vel $a = 0$, vel $a + nb = 0$; i.e. vel evanescat primus, vel ultimus resultantis æquationis terminus: & in genere ita assumi possint rationes inter a & b , ut quilibet numerus A fit radix resultantis æquationis, erit enim

$$\frac{b}{a} = \frac{A^n - pA^{n-1} + qA^{n-2} - rA^{n-3} + \&c.}{pA^{n-1} - 2qA^{n-2} + 3rA^{n-3} + \&c.} \bullet$$

Cor. 3. Ducantur termini datæ æquationis in (m) successivas arithmeticas series, & dilui possunt quicunque (m) termini datæ æquationis: multiplicatio enim arithmeticæ uniuscujusque seriei diluere potest quemlibet datæ æquationis terminum: fit enim terminus arithmeticæ seriei, qui in terminum æquationis diluendum ducatur, nihilo æqualis; & diluetur terminus datæ æquationis quæsitus; & sic de reliquis.

E

E. G.

E. G Sit æquatio, $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$, & requiratur eliminatio termini rx^{n-3} ; ducatur hæc æquatio in arithmeti-
cam seriem $-3b, -2b, -b, 0, b, 2b, 3b, \&c.$ cujus terminus sit (0) ,
qui ducatur in (rx^{n-3}) terminum datæ æquationis destruendum, & re-
sultabit æquatio quæsitæ $-3bx^n + 2bpx^{n-1} - bq x^{n-2} + bsx^{n-4} - \&c. = 0$,
cujus terminus (rx^{n-3}) deest, & cujus radices erunt limites inter
respectivas datæ æquationis radices.

Cor. 4. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$,
cujus radices sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$; radices æquationis $ax^n - a + bpx^{n-1}$
 $+ a + 2bqx^{n-2} - a + 3brx^{n-3} + \&c. = 0$ erunt limites inter α & β ,
 β & γ , γ & δ , δ & ϵ , &c. respective: ducantur hujus æquationis suc-
cessivi termini in terminos alterius arithmeticæ seriei $a', a' + b',$
 $a' + 2b', a' + 3b', \&c.$ successivos, & resultat æquatio

$a \times a'x^n - a + b \times a' + b'px^{n-1} + a + 2b \times a' + 2b'qx^{n-2} - a + 3b \times a' + 3b'rx^{n-3} + \&c. = 0.$
cujus radices erunt limites inter α & γ , β & δ , γ & ϵ , &c. & sic dein-
ceps; et similiter sit data æquatio $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + sx^{n-4} + \&c. = 0$.
 $Px^{n-1} + Qx^{n-2} + Rx^{n-3} + Sx^{n-4} + \&c. = 0$, cujus radices sint
 $\alpha, \beta, \gamma, \delta, \&c. \pi, \rho, \sigma, \tau, \&c.$ quarum α major sit quam β , β quam γ ,
 γ quam δ , &c. π quam ρ , ρ quam σ , &c. & inter radices α & π , β & ρ ,
 γ & σ , δ & τ , &c. interponantur $(m-1)$ radices: hujus æquationis du-
cantur termini successivi in terminos duarum arithmeticarum serie-
rum successivos respective

$$n, n-1, n-2, n-3, n-4, \&c.$$

$$0, 1, 2, 3, 4, \&c.$$

& resultabunt duæ æquationes $(n-1)$ dimensionum

$$nx^{n-1} + n-1 px^{n-2} + n-2 qx^{n-3} + n-3 rx^{n-4} + \&c. = 0;$$

$$\& + px^{n-1} + 2qx^{n-2} + 3rx^{n-3} + \&c. = 0.$$

quarum radices erunt limites respective inter radices α & β , β & γ ,
 γ & δ , δ & ϵ , &c. datæ æquationis.

Ducantur termini successivi duarum æquationum resultantium in
successivos terminos duarum prædictarum arithmeticarum serierum

$$n, n-1, n-2, n-3, \&c.$$

$$0, 1, 2, 3, \&c.$$

&

& resultabunt tres æquationes $(n-2)$ dimensionum

$$\begin{aligned} & n \times n-1 x^{n-2} + n-1 \times n-2 p x^{n-3} + n-2 \times n-3 q x^{n-4} + n-3 \times n-4 r x^{n-5} + \&c. = 0 \\ & \& + 1 \times n-1 p x^{n-2} + 2 \times n-2 q x^{n-3} + 3 \times n-3 r x^{n-4} + 4 \times n-4 s x^{n-5} + \&c. = 0 \\ & \& 1 \times 2 q x^{n-2} + 2 \times 3 r x^{n-3} + 3 \times 4 s x^{n-4} + 4 \times 5 t x^{n-5} + \&c. = 0 \end{aligned}$$

quarum radices erunt respective limites inter α & γ , β & δ , γ & ϵ , &c.

Et generaliter eodem prorsus modo invenientur $(m+1)$ æquationes $(n-m)$ dimensionum, $n \times n-1 \times n-2 \times \dots \times n-m+1 x^{n-m} + n-1 \times n-2 \times n-3 \dots n-m p x^{n-m-1} + n-2 \times n-3 \times n-4 \dots n-m-1 q x^{n-m-2} + n-3 \times n-4 \times n-5 \dots n-m-2 r x^{n-m-3} + n-4 \times n-5 \dots n-m-3 s x^{n-m-4} + \&c. = 0$, & $+ 1 \times n-1 \times n-2 \dots n-m+1 p x^{n-m} + 2 \times n-2 \times n-3 \dots n-m q x^{n-m-1} + 3 \times n-3 \times n-4 \dots n-m-1 r x^{n-m-2} + 4 \times n-4 \times n-5 \dots n-m-2 s x^{n-m-3} + 5 \times n-5 \times n-6 \dots n-m-3 t x^{n-m-4} + \&c. = 0$, & $1 \times 2 \times n-2 \times n-3 \dots n-m+1 q x^{n-m} + 2 \times 3 \times n-3 \times n-4 \dots n-m r x^{n-m-1} + 3 \times 4 \times n-4 \times n-5 \dots n-m-1 s x^{n-m-2} + \&c. = 0$; & in genere æquatio, cujus distantia e prima sit (r) , evadit $1.2.3 \dots r \times n-r \times n-r-1 \times n-r-2 \dots n-m+1 P x^{n-m} + 2.3.4 \dots r+1 \times n-r-1 \times n-r-2 \times n-r-3 \dots n-m Q x^{n-m-1} + 3.4.5 \dots r+2 \times n-r-2 \times n-r-3 \times n-r-4 \dots n-m-1 R x^{n-m-2} + 4.5.6 \dots r+3 \times n-r-3 \times n-r-4 \times n-r-5 \dots n-m-2 S x^{n-m-3} + 5 \times 6 \times 7 \dots r+4 \times n-r-4 \times n-r-5 \times n-r-6 \dots n-m-3 T x^{n-m-4} + \&c. = 0$, quarum $(m+1)$ æquationum radices erunt limites inter α & π , β & ρ , γ & σ , δ & τ , &c.

Cor. 5. Sit data æquatio $x^n - p x^{n-1} + q x^{n-2} - r x^{n-3} + \&c. = 0$: ducantur termini hujus æquationis successive in terminos arithmetice seriei $a, a+b, a+2b, a+3b, \&c.$: & si (m) radices sint (α) in datâ æquatione, $(m-1)$ radices etiam erunt (α) in resultanti æquatione: ducatur æquatio resultans in alteram arithmeticam seriem, & $(m-2)$ radices etiam erunt (α) in æquatione deductâ, & sic deinceps. Æquationes, quarum radices sint limites inter radices datæ æquationis, etiam inveniri possunt vel ducendo terminos datæ æquationis in terminos harmonicæ seriei, vel augendo, vel diminuendo, vel utcumque mutando datæ æquationis radices per quantitatem minorem quam differentiam inter ullas duas quascunque successive datæ æquationis radices.

Omnis regula, quæ generaliter inveniatur limites inter radices vel quantitates, necessario etiam inueniet, quando prædictæ radices vel quantitates fiunt æquales.

PROB. VII.

Datâ æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$, cujus radices sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ successive, datis etiam limitibus inter $\alpha \& \beta, \beta \& \gamma, \gamma \& \delta, \delta \& \epsilon, \&c.$ invenire numerum possibilium vel impossibilium radicum in datâ æquatione contentarum.

Sit π major quam maxima radix (α), ρ limes inter α & β , σ inter β & γ , τ inter γ & δ , &c.

Substituantur limites ($\pi, \rho, \sigma, \tau, \&c.$) pro incognitâ quantitate (x) in datâ æquatione, & tot erunt radices possibiles, quot sint mutationes signorum quantitatum resultantium de $+$ in $-$, & $-$ in $+$; cæteræ autem impossibiles sunt.

Hoc constat e theoremate secundo.

Cor. 1. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$, & radices æquationis $nx^{n-1} - (n-1)px^{n-2} + (n-2)qx^{n-3} - (n-3)rx^{n-4} + \&c. = 0$ erunt limites inter radices datæ æquationis: inveniantur radices hujus æquationis, & substituantur pro radice (x) in priori æquatione, & e numero mutationum signorum de $+$ in $-$, & $-$ in $+$ quantitatum resultantium constabit possibilium & impossibilium radicum numerus.

Cor. 2. Hinc ab extractione radicum æquationis $(n-1)$ dimensionum $nx^{n-1} - (n-1)px^{n-2} + (n-2)qx^{n-3} - (n-3)rx^{n-4} + \&c. = 0$; vel si modo deleatur secundus datæ æquationis terminus, i. e. fit æquatio $x^n + qx^{n-2} - rx^{n-3} + \&c. = 0$, ab extractione radicum æquationis $(n-2)$ dimensionum (in hoc enim casu unus limes inter radices datæ æquationis nihilo erit æqualis) erui potest impossibilium radicum numerus.

E. G. Sit æquatio $x^2 + q = 0$, & limes inter ejus duas radices erit nihilo æqualis, & consequenter duæ radices erunt possibiles, necne, prout q sit negativa, vel affirmativa quantitas.

Ex. 2. Sit $x^3 + qx - r = 0$, cujus radices sint α, β, γ , successive; ducantur

ducantur successivi hujus æquationis termini in successivos arithmetice seriei 0, 1, 2, 3 terminos, & resultat simplex æquatio $2qx - 3r = 0$; cujus radix $\left(\frac{3r}{2q}\right)$ scribatur pro incognitâ quantitate (x) in datâ æquatione, & resultat $\frac{27r^3 + 4rq^3}{8q^3}$; scribatur etiam (0) pro incognitâ quantitate in datâ æquatione, & resultat $-r$: 1^{mo} fit $\frac{r}{q}$ affirmativa quantitas, & $\frac{3r}{2q}$ erit limes inter α & β , (0) vero limes inter β & γ ; & tres radices datæ æquationis erunt possibiles, necne, prout $\frac{27r^3 + 4rq^3}{8q^3}$, vel quod semper idem erit, $27r^3 + 4q^3$ fit negativa, vel affirmativa quantitas: 2^{do} fit $\frac{r}{q}$ negativa quantitas, & consequenter (0) limes inter α & β , & $\frac{3r}{2q}$ inter β & γ ; & tres radices datæ æquationis erunt possibiles, necne, prout $\frac{27r^3 + 4rq^3}{8q^3}$ fit affirmativa vel negativa quantitas, *i. e.* prout $27r^3 + 4q^3$ fit negativa vel affirmativa quantitas.

Ex. 3. Sit æquatio $x^4 + qx^3 - rx + s = 0$; & æquatio, cujus radices sint limites inter radices datæ æquationis, erit $2qx^3 - 3rx^2 + 4sx = 0$, cujus tres radices erunt 0, $\frac{3r \pm \sqrt{9r^2 - 32sq}}{4q}$.

1^{mo}. Sit $9r^2 - 32sq$ negativa quantitas; & quatuor, vel duæ datæ æquationis radices erunt impossibiles, prout s fit affirmativa, vel negativa quantitas.

2^{do}. Sit $9r^2 - 32sq$ affirmativa quantitas, & s & q diversa habeant signa; & substituantur pro (x) in datâ æquatione $\frac{3r + \sqrt{9r^2 - 32sq}}{4q}$,

0, $\frac{3r - \sqrt{9r^2 - 32sq}}{4q}$, & si quantitatum resultantium prima fit ne-

gativa,

gativa, secunda vero affirmativa, tertia autem negativa; tum omnes datæ æquationis radices erunt possibiles; sin vero quantitates resultantes omnes fuerint affirmativæ, tum nullam habebit possibilem radicem data æquatio; aliter vero duas.

Si s & q eadem habeant signa, & r sit negativa quantitas; substitu-
antur quantitates o , $\frac{-3r + \sqrt{9r^2 - 32sq}}{4q}$, $\frac{-3r - \sqrt{9r^2 - 32sq}}{4q}$

successive; si vero r sit affirmativa quantitas, substituantur o ,
 $\frac{3r - \sqrt{9r^2 - 32sq}}{4q}$, $\frac{3r + \sqrt{9r^2 - 32sq}}{4q}$ successive pro radice (x) in

datâ æquatione; & si quantitates resultantes respective habeant signa
 $-$, $+$, $-$; tum quatuor radices possibiles habet data æquatio: si
vero prædicta signa sint $+$, $+$, $+$; nullam possibilem radicem ha-
bet data æquatio, sin aliter duas.

Et sic ad æquationes majorum dimensionum progredi licet.

Cor. I. Quot impossibiles radices habet æquatio $n x^{n-1} - n-1 p x^{n-2}$
 $+ n-2 q x^{n-3} - \&c. = 0$; tot vel plures habet etiam æquatio
 $x^n - p x^{n-1} + q x^{n-2} - r x^{n-3} + \&c. = 0$.

Hoc constat e demonstratione ad problema sextum.

Cor. Datâ æquatione $x^n + p x^{n-1} + q x^{n-2} + r x^{n-3} + \&c. + P x^{n-m}$
 $+ Q x^{n-m-1} + R x^{n-m-2} + S x^{n-m-3} + T x^{n-m-4} + \&c. = 0$, cujus radi-
ces sint $\alpha, \beta, \gamma, \delta, \&c. \nu, \pi, \rho, \sigma, \tau$; quarum α sit maxima, & τ minima
radix, & β major quam γ , γ quam δ , &c. ν quam π , π quam ρ , ρ
quam σ , &c. constat e Corollario 4^{to} Prob. sexti tres esse æquationes
 $n-2$ dimensionum

$$\begin{aligned} n \times n-1 x^{n-2} + n-1 \times n-2 p x^{n-3} + n-2 \times n-3 q x^{n-4} + n-3 \times n-4 r x^{n-5} + \&c. &= 0 \\ \& 1 \times n-1 p x^{n-2} + 2 \times n-2 q x^{n-3} + 3 \times n-3 r x^{n-4} + 4 \times n-4 s x^{n-5} + \&c. &= 0 \\ \& 1 \times 2 q x^{n-2} + 2 \times 3 r x^{n-3} + 3 \times 4 s x^{n-4} + \&c. &= 0 \end{aligned}$$

quarum radices erunt respective limites inter α & γ , β & δ , γ & ϵ , &c.;
at forsan nullum habeant limitem hæ tres æquationes inter quascun-
que duas successive radices E. G. (α & β); possit enim maxima radix
harum trium æquationum esse limes inter β & γ ; & consequenter re-
solutio harum trium æquationum resultantium haud necessario in-
venit numerum impossibilium radicum in datâ æquatione contenta-
rum

rum; & a fortiori resolutio nullius inferioris æquationis semper verum præbebit numerum datæ æquationis radicum. E. G. Si per Corollarium tertium Prob. 6. diluantur $(n-2)$ termini datæ æquationis, & consequenter $(n-1)$ quadraticæ æquationes $n \times n-1 x^2 + 2 \times n-1 p x + 2 q = 0$, $n-1 \times n-2 p x^2 + 2 \times 2 \times n-2 q x + 2 \cdot 3 r = 0$, $n-2 \times n-3 q x^2 + 2 \times 3 \times n-3 r x + 3 \cdot 4 s = 0$, &c. & generaliter quadratica æquatio, cujus distantia a primâ fit m , erit $n-m \times n-m-1 \times P x^2 + 2 \times m+1 \times n-m-1 Q x + m+1 \times m+2 R = 0$, quarum omnium radices erunt limites inter α & σ , & β & τ ; & exinde e resolutione harum omnium æquationum frequentur haud inveniri potest impossibilium radicum numerus, plures enim harum æquationum radices possint esse limites inter easdem duas datæ æquationis radices.

Diluantur omnes præter quatuor successivos datæ æquationes terminos, & resultabunt $(n-2)$ cubicæ æquationes $n \times n-1 \times n-2 x^3 + 3 \times n-1 \times n-2 p x^2 + 3 \times 2 \times n-2 q x + 1 \times 2 \times 3 r = 0$, $n-1 \times n-2 \times n-3 p x^3 + 3 \times 2 \times n-2 \times n-3 q x^2 + 3 \times 2 \times 3 \times n-3 r x + 2 \times 3 \times 4 s = 0$, & generaliter cubica æquatio, cujus distantia a primâ fit (m) erit $n-m \times n-m-1 \times n-m-2 \times P x^3 + 3 \times m+1 \times n-m-1 \times n-m-2 Q x^2 + 3 \times m+1 \times m+2 \times n-m-2 R x + m+1 \times m+2 \times m+3 S = 0$; harum $(n-2)$ æquationum cubicarum radices erunt limites inter α & ρ , β & σ , γ & τ ; & consequenter ex earum resolutione haud generaliter invenitur impossibilium radicum numerus: & sic de omnibus æquationibus, in quibus haud contineantur limites inter α & β , β & γ , γ & δ , &c.

P R O B. VIII.

Invenire limites inter α & β , β & γ , γ & δ , &c.

Transformetur per Exem. 6. Prob. 5. data æquatio in alteram, cujus radices sint $\frac{1}{\alpha-\beta}$, $\frac{1}{\alpha-\gamma}$, $\frac{1}{\alpha-\delta}$, $\frac{1}{\beta-\gamma}$, $\frac{1}{\beta-\delta}$, $\frac{1}{\gamma-\delta}$, &c. respective; fit $\frac{1}{A}$ æqualis, vel major quam maxima resultantis æquationis radix, & erit A minor quam differentia, vel inter α & β , β & γ , γ & δ , &c.

&c

& sit S æqualis vel major quam maxima datæ æquationis radix.

Inventâ igitur quantitate S , quæ major est quam maxima radix (a); & A , quæ æqualis vel minor erit quam minima differentia inter ullas duas successivas radices; continuo auferantur A , $2A$, $3A$, $4A$, $5A$, &c. respectively de S ; & tandem resultabunt limites inter successivas radices α & β , β & γ , γ & δ , &c.

Scribatur pro x in data æquatione $x + a$, & $x + b$ successive; & e mutatione signorum facile dici possit, quot radices vel possibiles vel impossibiles inter quantitates a & b contineantur.

P R O B. IX.

1. Sint $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, &c. quæcunque quantitates, quarum numerus sit (n), & sint $2a, 2b, 2c, 2d, 2e, 2f$, &c. quicunque indices pares numeri, & quorum numerus sit (m), & quorum a major sit quam b , b quam c , c quam d , &c. sint etiam p, q, r, s, t , &c. quicunque indices, quorum numerus sit (M), & sit p æqualis vel major quam q , q quam r , r quam s , s quam t , &c. & sit etiam $2a$ æqualis vel major quam p , $2b$ æqualis vel major quam q , ni $2a$ æqualis vel major sit quam $p+q$; $2c$ æqualis vel major quam r , ni $2a+2b$, vel æqualis vel major sit quam $p+q+r$; & sic deinceps; & sit $2a+2b+2c+2d+2e+\&c.=p+q+r+s+t+\&c.$

Et summa omnium hujuscemodi contentorum $\alpha^{2a} \beta^{2b} \gamma^{2c} \delta^{2d} \epsilon^{2e} \&c. + \alpha^{2a} \beta^{2b} \gamma^{2c} \delta^{2d} \zeta^{2e} \&c. + \beta^{2a} \gamma^{2b} \delta^{2c} \epsilon^{2d} \zeta^{2e} \&c. + \&c. = P$ habet vel eandem, vel majorem rationem ad summam contentorum $\alpha^p \beta^q \gamma^r \delta^s \epsilon^t \&c. + \alpha^p \beta^q \gamma^r \delta^s \zeta^t \&c. + \beta^p \gamma^q \delta^r \epsilon^s \zeta^t \&c. + \&c. = Q$, quam numerus contentorum in priori summâ ad numerum contentorum in posteriori; i. e. quam $n \times n-1 \times n-2 \times n-3 \dots n-m+1$ ad $n \times n-1 \times n-2 \times n-3 \dots n-M+1$: vel $1 : n-m \times n-m-1 \times n-m-2 \dots n-M+1$.

Hoc problema, scilicet quod $n-m \times n-m-1 \times n-m-2 \dots n-M+1 \times P$ æqualis vel major sit quam Q , sequitur ex hâc ratione, nempe quantitas $n-m \times n-m-1 \times n-m-2 \dots n-M+1 \times P - Q$ semper probari potest æqualis aggregato quadratorum e compluribus quantitatibus

tibus. Hæc enim solummodo est methodus, e quâ concludi potest datam quantitatem semper esse affirmativam.

Si (*b*) indices in priori quantitate (*P*) sint *a*, *k* indices vero *b*, *l* & *o* indices respective *c* & *d*, &c.; & sint etiam in posteriori quantitate (*Q*) (*H*) indices *p*, *K* indices *q*, *L* & *O* indices respective *r* & *s*, &c. & $1 \times 2 \times 3 \times \dots b \times 1 \times 2 \times 3 \times \dots k \times 1 \times 2 \times 3 \times \dots l \times 1 \times 2 \times 3 \times \dots o \times \&c. \times P : 1 \times 2 \times 3 \times \dots H \times 1 \times 2 \times 3 \times \dots K \times 1 \times 2 \times 3 \times \dots L \times 1 \times 2 \times 3 \times \dots O \times \&c. \times Q$ habet vel eandem vel majorem rationem, quam $1 : n - m \times n - m - 1 \times n - m - 2 \times \dots n - M + 1$.

Cor. 1. Sit æquatio $x^n + p x^{n-1} + q x^{n-2} + r x^{n-3} + s x^{n-4} + \&c. \dots + g x^{n-m+l+5} + f x^{n-m+l+4} + e x^{n-m+l+3} + d x^{n-m+l+2} + c x^{n-m+l+1} + b x^{n-m+l} + a x^{n-m} + \&c. \dots K x^{n-m+5} + L x^{n-m+4} + M x^{n-m+3} + N x^{n-m+2} + O x^{n-m+1} + P x^{n-m} + Q x^{n-m-1} + R x^{n-m-2} + S x^{n-m-3} + T x^{n-m-4} + V x^{n-m-5} + \&c. \dots A x^{n-m-l} + B x^{n-m-l-1} + C x^{n-m-l-2} + D x^{n-m-l-3} + E x^{n-m-l-4} + F x^{n-m-l-5} + \&c. = 0$; cujus radices sint omnes possibiles, & per $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \&c.$ respective designatæ.

Per Theorema Summa (π) omnium hujuscemodi contentorum $\alpha^2 \beta^2 \gamma^2 \delta^2 \epsilon^2 \zeta^2 \&c. + \alpha^2 \beta^2 \gamma^2 \delta^2 \zeta^2 \&c. + \alpha^2 \beta^2 \gamma^2 \epsilon^2 \zeta^2 \&c. + \&c. + \beta^2 \gamma^2 \delta^2 \epsilon^2 \zeta^2 \&c. + \&c.$ (in quibus singulis contentis numerus literarum $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ contentarum sit m) multiplicata in contentum $m \times m - 1 \times m - 2 \times m - 3 \times \dots m - l + 1 \times n - m \times n - m - 1 \times n - m - 2 \times \dots n - m - l + 1 = X$ æqualis vel major erit quam summa (σ) omnium contentorum $\alpha^2 \beta^2 \gamma^2 \&c. \times \epsilon \zeta \&c. + \alpha^2 \beta^2 \delta^2 \&c. \times \gamma \zeta \&c. + \beta^2 \gamma^2 \delta^2 \&c. \times \alpha \epsilon \&c. + \beta^2 \gamma^2 \epsilon^2 \&c. \times \alpha \zeta \&c. + \&c.$ (in singulis hisce contentis numerus literarum, in quibus duæ inveniuntur dimensiones, sit $m - l$; numerus autem literarum, in quibus una solummodo continetur litera, erit $2l$) multiplicata in contentum $1 \times 2 \times 3 \times 4 \times \dots 2l = Z$.

Per Exemplum 2. Problem. 4^{ti}. $\pi = P^2 - 2QO + 2RN - 2SM + 2TL - 2VK + \&c.$ & $\sigma = Aa - \frac{2l+2}{2} Bb + \frac{2l+1}{2} \times \frac{2l+4}{2} Cc - \frac{2l+1}{2} \times \frac{2l+2}{2} \times \frac{2l+6}{3} Dd + \frac{2l+1}{2} \times \frac{2l+2}{2} \times \frac{2l+3}{3} \times \frac{2l+8}{4} Ee - \&c.$
F ducantur

fit etiamque a vel æqualis vel major quam p, b vel æqualis vel major quam q, ni a vel æqualis vel major sit quam p+q; c vel æqualis vel major quam r, ni a+b vel æqualis vel major sit quam p+q+r; d vel æqualis vel major quam s, ni a+b+c vel æqualis vel major sit quam p+q+r+s; &c. sic deinceps; & a+b+c+d+e+&c. = p+q+r+s+t+&c.

Et summa (P) omnium hujusmodi contentorum $\alpha^a \beta^b \gamma^c \delta^d \epsilon^e$ &c. + $\alpha^b \beta^a \gamma^c \delta^d \epsilon^e$ &c. + $\alpha^a \beta^b \gamma^c \delta^d \zeta^f$ + &c. habet vel eandem vel majorem rationem ad summam (Q) contentorum $\alpha^a \beta^a \gamma^c \delta^d \epsilon^e$ &c. + $\alpha^a \beta^b \gamma^c \delta^d \epsilon^e$ &c. + $\alpha^a \beta^b \gamma^c \delta^d \zeta^f$ &c. + &c. quam numerus contentorum in priori summâ ad numerum contentorum in posteriori, i. e. quam 1 : $\frac{n-m \times n-m+1 \times n-m+2 \times \dots \times n-M+1}{n-M+1}$.

Quantitas $\frac{n-m \times n-m+1 \times n-m+2 \dots n-M+1}{n-M+1} \times P$ semper probari potest esse major quam Q; ex eo, quod probari potest quantitatem $\frac{n-m \times n-m+1 \times n-m+2 \dots n-M+1}{n-M+1} \times P - Q$, æqualem esse aggregato quadratorum in diversa literalia contenta multiplicatorum.

Sint (b) indices in priori quantitate (P) æquales per (a) designati, k indices æquales per (b), l & o indices æquales per c & d respective designati, &c. Sint etiam b' indices æquales per (p) in posteriori quantitate designati, k' indices æquales per (q), l' & o' indices æquales per r & s respective designati, &c. & $1.2.3 \dots b \times 1.2.3 \dots k \times 1.2.3 \dots l \times 1.2.3 \dots o \times \&c. \times P : 1 \times 2 \times 3 \dots b' \times 1 \times 2 \times 3 \dots k' \times 1.2.3 \dots l' \times 1.2.3 \dots o' \times Q$ habet æqualem vel majorem rationem quam $1 : \frac{n-m+1 \times n-m+2 \dots n-M+1}{n-M+1}$.

Cor. Sit æquatio $x^n + p x^{n-1} + q x^{n-2} + r x^{n-3} + \&c. + g x^{n-l+6} + f x^{n-l+5} + e x^{n-l+4} + d x^{n-l+3} + c x^{n-l+2} + b x^{n-l+1} + a x^{n-1} + \&c. + K x^{n-m+5} + L x^{n-m+4} + M x^{n-m+3} + N x^{n-m+2} + O x^{n-m+1} + P x^{n-m} + Q x^{n-m-1} + R x^{n-m-2} + S x^{n-m-3} + T x^{n-m-4} + \&c. + A x^{n-m-r} + B x^{n-m-r-1} + C x^{n-m-r-2} + D x^{n-m-r-3} + E x^{n-m-r-4} + F x^{n-m-r-5} + G x^{n-m-r-6} + \&c. + h x^{n-2m+l-r} + i x^{n-2m+l-r-1} + k x^{n-2m+l-r-2} + l x^{n-2m+l-r-3} + \&c. = 0$,
cujus radices sint omnes possibiles, & respective $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ &

sint omnes hæ radices vel affirmativæ vel negativæ.

Per Theorema Summa (π) omnium hujusmodi contentorum $\alpha^2 \beta^2 \gamma^2 \&c. \times \delta^2 \&c. + \alpha^2 \gamma^2 \delta^2 \&c. \times \beta^2 \&c. + \alpha^2 \beta^2 \gamma^2 \&c. \times \delta^2 \&c. + \&c.$ (in quibus singulis contentis numerus literarum, quæ duas habeant dimensiones, sit m ; numerus autem literarum, quæ unam solummodo habet dimensionem, sit r) vel eandem vel majorem rationem habet ad summam (σ) omnium contentorum hujusmodi $\alpha^2 \beta^2 \&c. \times \gamma^2 \delta^2 \&c. + \alpha^2 \gamma^2 \&c. \times \beta^2 \delta^2 \&c. + \beta^2 \gamma^2 \&c. \times \alpha^2 \delta^2 \&c. + \&c.$ (in quibus singulis contentis numerus literarum, quæ duas habent dimensiones, sit l ; numerus autem literarum, quæ unam solummodo habet dimensionem, erit $2m+r-2l$) quam $\overline{r+1} \times \overline{r+2} \times \overline{r+3} \times \dots \times \overline{2m-2l+r} = Z$; $\overline{l+1} \times \overline{l+2} \times \overline{l+3} \times \dots \times \overline{m} \times \overline{n-m-r} \times \overline{n-m-r-1} \times \overline{n-m-r-2} \dots \overline{n-2m+l-r+1} = X$; & consequenter $\overline{l+1} \times \overline{l+2} \dots \overline{m} \times \overline{n-m-r} \times \overline{n-m-r-1} \dots \overline{n-2m+l-r+1} \times \pi$ vel æqualis vel major erit quam $\overline{r+1} \times \overline{r+2} \dots \overline{2m-2l+r} \times \sigma$.

Si vero $m = l+1$, tum erit $m \times n-m-r \times \pi$ æqualis vel major quam $\overline{r+1} \times \overline{r+2} \times \sigma$.

Per Exemplum 2. Problem 4^{ti}.

$$\pi = PA - \overline{r+2} OB + \overline{r+1} \times \frac{\overline{r+4}}{2} NC - \overline{r+1} \times \frac{\overline{r+2}}{2} \times \frac{\overline{r+6}}{3} MD + \&c.$$

$$\& \sigma = ba - \overline{2m-2l+r+2} ib + \overline{2m-2l+r+1} \times \frac{\overline{2m-2l+r+4}}{2} kc -$$

$$\overline{2m-2l+r+1} \times \frac{\overline{2m-2l+r+2}}{2} \times \frac{\overline{2m-2l+r+6}}{3} ld + \&c.$$

Ducantur hæ quantitates respective in X & Z ; & resultant $X\pi =$

$$XPA - \overline{r+2} XOB + \overline{r+1} \times \frac{\overline{r+4}}{2} XNC - \overline{r+1} \times \frac{\overline{r+2}}{2} \times \frac{\overline{r+6}}{3} X$$

$$MD + \&c. \& Z\sigma = Zba - \overline{2m-2l+r+2} Zib + \overline{2m-2l+r+1} \times$$

$$\frac{\overline{2m-2l+r+4}}{2} Zkc - \overline{2m-2l+r+1} \times \frac{\overline{2m-2l+r+2}}{2} \times \frac{\overline{2m-2l+r+6}}{3} Zld + \&c.$$

cujus

cujus differentia erit $XPA - \overline{r+2} XOB + \overline{r+1} \times \frac{r+4}{2} \times XNC - \&c.$

ad $m-l$ terminos $- Z \pm \frac{\overline{r+1} \times \overline{r+2} \times \dots \times \overline{r+m-l-1} \times \overline{r+2m-2l} \times X}{1.2.3 \dots m-l} \times$

$ba + \overline{2m-2l+r+2} Z \pm \frac{\overline{r+1} \times \overline{r+2} \times \dots \times \overline{r+m-l} \times \overline{r+2m-2l+2} \times X}{1.2.3.4 \dots m-l+1} \times lb +$

$\frac{\overline{2m-2l+r+1} \times \overline{2m-2l+r+4} Z \pm \frac{\overline{r+1} \times \overline{r+2} \times \overline{r+3} \times \dots \times \overline{r+m-l+1} \times \overline{r+2m-2l+4} \times X}{1.2.3.4 \dots m-l+2}}$

$\times kc + \&c.$

Si $m-l$ sit par numerus, tum signum quantitati

$\left(\frac{\overline{r+1} \times \overline{r+2} \times \dots \times \overline{r+m-l-1} \times \overline{r+2m-2l} \times X}{1.2.3 \dots m-l} \right)$ affixum erit negativum;

sin aliter affirmativum.

Sit $m=l+1$, & X & Z fiunt respective $\overline{m \times n - m - r}$, & $\overline{r+1} \times \overline{r+2}$; fit etiam r vel nihil, vel 2, vel 4, vel 6, vel 8, &c.; & Px^{n-m} & Ax^{n-m-r} iidem fiunt termini cum r sit 0: & exinde resultat quantitas, $XP^2 - 2 \times \overline{X+1} \times \overline{2O} + 2 \times \overline{X+4} \times \overline{RN} - 2 \times \overline{X+9} \times \overline{SM} + 2 \times \overline{X+16} \times \overline{TL} - 2 \times \overline{X+25} \times \overline{VK} + \&c. = a'$: Hæc quantitas semper erit affirmativa, cum radices $\alpha, \beta, \gamma, \delta$, &c. sint possibiles.

Sit $r=2$, tum X & Z erunt respective $\overline{n-m-1} \times \overline{m-1}$, & 3×4 ; & quantitas quæsitæ $X \overline{2O} - 4 \times \overline{X+3} \times \overline{RN} + 9 \times \overline{X+8} \times \overline{SM} - 16 \times \overline{X+15} \times \overline{TL} + 25 \times \overline{X+24} \times \overline{VK} - \&c. = b'$.

Sit $r=4$, tum X & Z erunt respective $\overline{m-2} \times \overline{n-m-2}$, & $5 \times 6 = 30$; & resultat $X \times \overline{RN} - 6 \times \overline{X+5} \times \overline{SM} + 20 \times \overline{X+12} \times \overline{TL} - 50 \times \overline{X+21} \times \overline{VK} + \&c. = c'$, & sic deinceps.

Quantitates a', b', c', d', e' , &c. resultantes semper erunt affirmativæ, cum radices $\alpha, \beta, \gamma, \delta, \epsilon$, &c. sint affirmativæ.

Ex his quantitatibus inveniri possunt quantitates $\overline{m \times n - m} P^2 - 2 \overline{2O}$, & $\overline{m-1} \times \overline{n-m-1} \overline{2O} - 12 \overline{RN}$, & $\overline{m-2} \times \overline{n-m-2} \overline{RN} - 30 \overline{SM}$, & $\overline{m-3} \times \overline{n-m-3} \overline{SM} - 56 \overline{VK}$, & sic deinceps; quæ omnes erunt

erunt aggregata ex hisce quantitativibus $a', b', c', d', e', \&c.$ in affirmativos numeros ductis.

PROB. IX.

E theorematibus præcedentibus plures invenire regulas, e quibus deduci possunt impossibiles radices.

Sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ quæcunque impossibiles radices, & ex iis per duo theoremata præcedentia deduci possit series quantitatum $a, b, c, d, e, f, \&c.$ in quibus similiter inter se involvuntur radices $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ & quæ semper erunt affirmativæ, cum radices $\alpha, \beta, \gamma, \delta, \&c.$ sint possibiles, & quæ secundum dimensiones radicum $\alpha, \beta, \gamma, \delta, \&c.$ regulariter ascendunt, vel descendunt.

Sit numerus radicum $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \&c. = n$, & numerus quantitatum $a, b, c, d, e, \&c. = n-1$; & scribantur quantitates $a, b, c, d, e, \&c.$ hoc modo $1 + a + b + c + d + e + \&c. + 1$; & tot vel plures erunt radices impossibiles, quot sint mutationes signorum terminorum de $+$ in $-$, & $-$ in $+$.

Ex. I. Sit æquatio, $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + sx^{n-4} + \&c. \dots$
 $Px^{n-m+1} + Qx^{n-m} + Rx^{n-m-1} + \&c. = 0$, cujus radices sint $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \iota, \&c.$ per theoremata præcedentia, (si modo radices sint possibiles,) $\frac{n-1}{2n} p^2$ major erit quam q ; $\frac{2 \times n-2}{3 \times n-3}$ major erit quam pr ;

& generaliter $\frac{m \times n - m}{m + 1 \times n - m + 1} \times Q^2$ major erit quam PR .

Ergo e radicibus $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \&c.$ invenietur series quantitatum $\frac{n-1}{2n} p^2 - q, \frac{2 \times n-2}{3 \times n-1} q^2 - pr, \frac{3 \times n-3}{4 \times n-2} r^2 - qs, \&c.$ quæ semper erunt affirmativæ quantitates, cum radices sint possibiles; & exinde tot vel plures radices erunt impossibiles, quot sint mutationes signorum quantitatum in datâ serie contentarum de $+$ in $-$, & $-$ in $+$; i. e. sub primo & ultimo datæ æquationis termino $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + sx^{n-4} + \&c. = 0$ colloca signum $+$; sub quolibet mediorum ter-

terminorum, si ejus quadratum ductum in prædictam fractionem majus sit quam rectangulum terminorum utrinque consistentium, colloca signum +, sin minus signum —; i. e. si $\frac{n-1}{2n} p^2$ majus sit quam q , colloca signum + sub termino px^{n-1} ; & eodem modo, si $\frac{2n-2}{3n-1} q^2$ majus sit quam pr , & $\frac{3n-3}{4n-2} r^2$ majus quam qs , colloca sub respectivis terminis qx^{n-2} , px^{n-3} , signum +; sin minus, signum —; & sic de reliquis: & tot vel plures radices erunt impossibiles, quot sint mutationes signorum resultantium de + in —, & — in +.

In hac & consimilibus regulis, ubi termini duo vel plures defint, sub primo terminorum deficientium collocandum est signum —, sub secundo signum +, sub tertio signum —, & sic deinceps, nisi quod sub ultimo terminorum simul deficientium semper collocandum est signum +, ubi termini deficientibus utrinque proximi habeant signa contraria.

Hæc methodus in quadraticis æquationibus verum præbet numerum impossibilium radicum: in cubicis autem probabilitatem inveniendi impossibiles radices non videtur majorem habere rationem ad probabilitatem fallendi quam 2 : 1. In æquationibus autem multo superiorum dimensionum verum impossibilium radicum numerum perraro deteget.

Quod $\frac{m \times n - m}{m + 1 \times n - m + 1} Q^2$ majus erit quam PR , cum radices datæ

æquationis sint possibiles, hac methodo probari potest. Notum est $Q = \alpha \beta \gamma \delta \epsilon$, &c. + $\alpha \beta \gamma \delta \zeta$ &c. + $\alpha \beta \gamma \epsilon \zeta$ &c. + &c. Et sit R summa quadratorum e singulis differentiis inter quoscunque duos coefficientis Q terminos.

Sit A summa ex omnibus prædictis quadratis, quorum duo termini in radice contenti a se invicem duas habent literas diversas.

Sint C & D , &c. summa omnium quadratorum, quorum termini a se invicem respective habent tres, vel quatuor, &c. diversas habent literas,

literas, i. e. sint $Y = \overline{a\beta\gamma\delta\epsilon\zeta\eta\theta\iota\kappa\lambda\mu\nu\omega\pi\rho\sigma\tau\upsilon\phi\chi\psi\omega\pi\rho\sigma\tau\upsilon\phi\chi\psi}$
 $+ \overline{a\beta\gamma\delta\epsilon\zeta\eta\theta\iota\kappa\lambda\mu\nu\omega\pi\rho\sigma\tau\upsilon\phi\chi\psi}$
 $A = \overline{a\beta\gamma\delta\epsilon\zeta\eta\theta\iota\kappa\lambda\mu\nu\omega\pi\rho\sigma\tau\upsilon\phi\chi\psi}$
 $B = \overline{a\beta\gamma\delta\epsilon\zeta\eta\theta\iota\kappa\lambda\mu\nu\omega\pi\rho\sigma\tau\upsilon\phi\chi\psi}$
 $C = \overline{a\beta\gamma\delta\epsilon\zeta\eta\theta\iota\kappa\lambda\mu\nu\omega\pi\rho\sigma\tau\upsilon\phi\chi\psi}$
 $D = \overline{a\beta\gamma\delta\epsilon\zeta\eta\theta\iota\kappa\lambda\mu\nu\omega\pi\rho\sigma\tau\upsilon\phi\chi\psi}$
 $\& \frac{m \times n - m}{m+1 \times n - m+1} Q$ major erit quam PR differentiâ $\frac{n+1 \times Y}{m+1 \times n - m+1}$
 $-\frac{1}{2}A - \frac{1}{2}B - \frac{1}{2}C - \frac{1}{2}D - \&c.$ ad $m+1$ terminos, quod facile
 e substitutione constabit.

Ex. 2. Sit æquatio $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + sx^{n-4} - \dots - Px^{n-m}$
 $+ Qx^{n-m-1} + Rx^{n-m-2} + Sx^{n-m-3} + Tx^{n-m-4} + \&c. = 0$. Colloca sub
 primo & ultimo termino signum +; sub secundo datæ æquationis
 termino px^{n-1} colloca signum +, si $\frac{9r^2}{n-1 \cdot n-2} - \frac{18pqr}{n \cdot n-1 \cdot n-2} -$

$\frac{3p^2q^2}{n^2 \cdot n-1} + \frac{6rp^3}{n^2 \cdot n-2} + \frac{8q^3}{n \cdot n-1}$, sit negativa quantitas, sin minus sig-
 num —; colloca sub tertio datæ æquationis termino qx^{n-2} signum +,
 si $\frac{4s^2p^2}{n-2 \cdot n-3} - \frac{12qrs p}{n-1 \cdot n-2 \cdot n-3} - \frac{3q^2r^2}{n-1^2 \cdot n-2} + \frac{16sq^3}{3 \times n-1^2 \cdot n-3}$
 $+ \frac{12r^3p}{n-1 \cdot n-2}$ sit negativa quantitas, sin minus signum —; & ge-

neraliter sub quolibet termino Qx^{n-m} colloca signum +, si
 $\frac{m+2 \cdot m+3^2 S^2 P^2}{n-m-1 \cdot n-m-2^2} - \frac{6 \times m+1 \cdot m+2 \cdot m+3 Q R S P}{n-m \cdot n-m-1 \cdot n-m-2} - \frac{3 \cdot m+1^2 \cdot m+2 Q^2 R^2}{n-m^2 \cdot n-m-1}$
 $+ \frac{4 \cdot m+1^2 \cdot m+3 S Q^3}{n-m^2 \cdot n-m-2} + \frac{4 \cdot m+1^2 \cdot m+2 R^3 P}{n-m \cdot n-m-1^2}$ sit negativa quantitas, sin
 minus signum —; & tot vel plures erunt radices impossibiles, quot
 sint

sint mutationes signorum de + in —, & — in +.

Hæc methodus semper impossibiles radices deteget, quando præcedens regula eas inveniet; & sæpe impossibiles radices inveniet, quando eadem fallit. E. G. In cubicis æquationibus impossibiles, si modo ullæ sint, radices semper deteget; in æquationibus $2n$ dimensionum, quarum omnes radices sint impossibiles, probabilitas verum impossibilium radicum numerum e præcedente regulâ detegendi videtur esse ad probabilitatem verum impossibilium radicum numerum ex hâc regulâ detegendi in ratione $2^n : 3^n$.

In æquationibus multo superiorum dimensionum hæc regula per raro verum impossibilium radicum numerum deteget.

Hæc regula semper inveniet, utrum ulla sit impossibilitas in quatuor primis datæ æquationis terminis, necne.

Et sic progredi liceat ad regulas, quæ inveniant, cum ulla impossibilitas contineatur in quintis, sextis, &c. primis terminis; & exinde constant diversæ regulæ de inveniendis impossibilium radicum numero.

Sæpe augendo vel diminuendo datæ æquationis radices per datas quantitates facile se manifestant impossibiles radices, quæ prius latere; & hoc constabit ex eo, quod, positâ radice impossibili esse $a + b\sqrt{-1}$, maxime pendet e ratione possibilis partis (a) ad impossibilem b , utrum hæ regulæ impossibilem radicem inveniant, necne: si vero a nihilo sit æqualis, tum semper detegent resultantes impossibiles radices omnes hujusce generis regulæ.

Ex. 3. Sit æquatio $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + sx^{n-4} + tx^{n-5} + wx^{n-6} + zx^{n-7} + \&c. \dots Ax^{n-m+4} + Bx^{n-m+3} + Cx^{n-m+2} + Dx^{n-m+1} + Ex^{n-m} + Fx^{n-m-1} + Gx^{n-m-2} + Hx^{n-m-3} + Ix^{n-m-4} + \&c. = 0$, cujus radices sint $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \&c.$

Colloca sub primo & ultimo datæ æquationis termino signum +, sub secundo signum +, si $\frac{n-1}{2n} p^2$ major sit quam q , sin minus signum —;

sub tertio signum +, si $\frac{n \times n - 1 - 2}{2n \cdot n - 1} q^2$ major sit quam $pr - s$,

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fin

fin minus signum —; colloca sub quarto termino rx^{n-3} signum +, si

$\frac{n \times n - 1 \times n - 2 - 2 \cdot 3}{2 \cdot n \cdot n - 1 \cdot n - 2} r^2$ major sit quam $qs - pt + v$; & colloca sub

quinto termino (sx^{n-4}) signum +, si $\frac{n \times n - 1 \times n - 2 \times n - 3 - 2 \cdot 3 \cdot 4}{2n \cdot n - 1 \cdot n - 2 \cdot n - 3} s^2$

major sit quam $rt - qv + pw - z$; & in genere colloca sub quolibet

termino (Ex^{n-m}) signum +, si $\frac{n \times n - 1 \times n - 2 \dots n - m + 1 - 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \dots m}{2 \cdot n \cdot n - 1 \cdot n - 2 \cdot n - 3 \dots n - m + 1} E^2$

major sit quam $DF - CG + BH - AI + \&c.$, fin minus signum —; & tot vel plures erunt impossibiles radices, quot resultant mutationes signorum de + in —, & — in +.

Facile e substitutione constabit $1 - \frac{1}{\frac{n \cdot n - 1}{2} \cdot \frac{n - 2}{3} \dots \frac{n - m + 1}{m}} \times E^2$ majorem

esse quam $2DF - 2CG + 2BH - 2AI + \&c.$ summâ $\frac{Y}{\frac{n \cdot n - 1}{2} \cdot \frac{n - 2}{3} \dots \frac{n - m + 1}{m}}$;

si modo Y sit summa quadratorum e singulis differentiis inter quoscunque duos coefficientis P terminos, i. e. fit $Y = \alpha\beta\gamma\delta\epsilon\&c. - \alpha\beta\gamma\delta\zeta\&c. + \alpha\beta\gamma\delta\epsilon\&c. - \alpha\beta\gamma\delta\eta\&c. + \beta\gamma\delta\epsilon\zeta\&c. - \beta\gamma\delta\eta\theta\&c. + \&c.$ ubi $E = \alpha\beta\gamma\delta\epsilon\&c. + \alpha\beta\gamma\delta\zeta\&c. + \alpha\beta\gamma\delta\eta\&c. + \beta\gamma\delta\epsilon\zeta\&c. + \beta\gamma\delta\eta\theta\&c. + \&c.$

Multæ regulæ hujusmodi etiam inveniri possunt e Corollar.

Theor. 3. exinde enim inveniri possunt diversæ quantitatum series, quæ semper erunt affirmativæ, quando radices sint possibiles.

E corollario Theorem. 4^{ti}. constant series quantitatum, quæ semper erunt affirmativæ, cum omnes datæ æquationis radices vel sint affirmativæ, vel negativæ; i. e. vel continuo mutantur signa datæ æquationis de + in —, & — in +, vel signa semper eadem sint.

Omnes hæ regulæ etiam generaliores reddi possint calculum instituendo de quantitibus superiorum dimensionum, quæ necessario erunt affirmativæ, quando radices sint possibiles: sed hæ methodi fient multo magis operosæ.

Hæc

Hæc principia etiam in alia problemata promovere licet. E. G. Sit Series quantitatum

$$\alpha + \beta + \gamma + \delta + \epsilon + \zeta + \&c. = p$$

$$\alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \gamma\delta + \alpha\epsilon + \&c. = q$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta + \alpha\gamma\epsilon + \&c. = r$$

$$\alpha\beta\gamma\delta + \alpha\beta\gamma\epsilon + \alpha\beta\gamma\zeta + \&c. = s$$

$$\alpha\beta\gamma\delta\epsilon + \alpha\beta\gamma\delta\zeta + \&c. = t$$

$$\&c. \quad \&c.$$

Hæ autem quantitates continuo regulariter ascendunt; & omnes erunt affirmativæ, si modo radices datæ æquationis sint omnes affirmativæ; ergo constat tot radices esse negativas vel impossibiles, quot sunt mutationes signorum de + in —, & — in + successive.

Collocentur termini hoc modo $1 + p + q + r + s + t + \&c.$ & tot vel plures erunt radices negativæ vel impossibiles, quot sint mutationes signorum de + in —, & — in +.

Collocentur termini autem subsequenti methodo $1 - p + q - r + s - t + \&c.$ & tot vel plures erunt negativæ vel impossibiles radices, quot sint continui progressus de + in +, vel — in —.

Eodem modo hæc ratiocinandi methodus in quamplurimos casus applicari potest: datâ enim serie quantitatum regulariter ascendentium $1 + a + b + c + d + e + f + \&c.$ quæ necessario fient affirmativæ, quando radices sint dati generis; sæpe e mutatione signorum de + in —, & — in + dici potest, quot sint quantitates diverfi generis.

Haud necessario in hâc argumentandi methodo, literæ $a, b, c, d, e, \&c.$ quantitates affirmativas vel negativas designent, sed etiam, cujuscunque assignabilis generis quantitates denotent.

P R O B. X.

E regulis hujusmodi datis alias invenire, quæ magis generalem præbeant solutionem.

Sit quæcunque æquatio

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$$

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ducatur

ducatur hæc æquatio in $x - a$, & resultat

$$\begin{array}{ccccccc} x^{n+1} & - & p x^n & + & q x^{n-1} & - & r x^{n-2} + s x^{n-3} - \&c. = 0. \\ & & - a & + & ap & - & aq & + ar \end{array}$$

Inveniatur, si modo facile possit, annon ita assumi possit incognita quantitas (a), ut præbeat datam impossibilium radicum notam, & exinde sequitur regula.

Hæc regula eodem modo adhuc magis generalis reddi possit, si modo resultans æquatio ducatur in $x - b$, & inveniatur, annon ita assumi possit quantitas b , ut præbeat notam impossibilium radicum e regulâ præcedente investigatam; & sic deinceps.

Ex. 1. Datâ æquatione $x^n - p x^{n-1} + q x^{n-2} - r x^{n-3} + \&c. + P x^{n-m+1} - Q x^{n-m} + R x^{n-m-1} - S x^{n-m-2} + \&c. = 0$, quæ ducta in $x - a$, fit $x^{n+1} - p x^n + q x^{n-1} - r x^{n-2} + \&c. - Q x^{n-m+1} + R x^{n-m} -$
 $- a + ap - aq - aP + aQ -$

$S x^{n-m-1} + \&c. = 0$; & quicumque assumatur incognitæ quantitas aR

tis (a) valor, sit $\overline{p+a^2}$ major quam $\overline{q+ap}$, $\overline{q+ap^2}$ major quam $\overline{p+a} \times \overline{r+aq}$, & sic deinceps.

Sed quoniam per hypothesin $\overline{p+a^2}$ semper major erit quam $\overline{q+ap}$; & $\overline{p+a^2} - ap = \overline{a+\frac{1}{2}p^2} + \frac{3}{2}p^2$ semper major erit quam q , quicumque sit valor quantitatis (a); & consequenter $\frac{3}{2}p^2$ major erit quam q ; & eodem modo invenitur $4 \times \overline{p^2} - q \times \overline{q^2} - pr$ major quam $\overline{pq-r^2}$; & generaliter $4 \times \overline{Q^2} - PR \times \overline{R^2} - SQ$ major invenitur quam $\overline{QR-SP^2}$.

Ex. 2. Sit æquatio data eadem ac in præcedente exemplo; ducatur etiam in $x - a$, & fit data regula ea, quæ continetur in Problem. IX.

Ex. 1.

Per datam regulam $\frac{n}{2 \times n + 1} \times \overline{p+a^2}$ semper major esse debet quam $\overline{q+ap}$, cum radices sint possibiles, quicumque sit valor incognitæ quantitatis (a), vel quod idem est $\frac{n}{2 \cdot n + 1} p^2 - \frac{1}{n+1} pa + \frac{n}{2 \times n + 1} a^2$ major quam q ; sed quoniam $\frac{n}{2 \cdot n + 1} a^2 - \frac{1}{n+1} pa + \frac{1}{2n \cdot n + 1} p^2$ est
 qua-

quadratum, & consequenter vel affirmativa quantitas vel nihil; ergo

$$\frac{n}{2 \cdot n + 1} p^2 - \frac{1}{2n \cdot n + 1} p^2 = \frac{n-1}{2n} p^2 \text{ major esse debet quam } q,$$

quod docet etiam prædicta regula: secundum autem eandem

regulam, si modo radices sint possibiles, $\frac{n-m \times m+1}{n-m+1 \times m+2} \times$

$2^2 a^2 + 2RQa + R^2$ semper major esse debet quam $a^2 PR + a \times PS + QR$

QS : in omni autem casu, quicumque sit valor incognitæ quantitatis

(a), minime hoc semper verum esse possit, ni $4 \times n-m \times m+1$ $2^2 -$

$n-m+1 \times m+2 PR \times n-m \times m+1 R^2 - n-m+1 \times m+2 QS$ major sit

quam $m n - m^2 - m - 2 QR - n-m+1 \times m+2 PS$. Hinc constat ex

hâc regula sæpe detegi posse impossibiles radices, cum nullæ e datâ regulâ colligi possint.

Eodem processu repetito regula adhuc magis generalis reddi possit, sed multo major operandi erit labor.

Transformatio æquationum sæpe detegit impossibiles datæ æquationis radices.

Transformari enim data potest æquatio in alias, quarum coefficientes vel earum quædam functiones ostendunt quasdam impossibiles in datâ æquatione latere radices.

Hujus rei exempla sequentes præbebunt transformationes.

THEOREM. 5.

1. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + \&c. = 0$ cujus radices sint $\alpha, \beta, \gamma, \delta, \&c.$ transformetur in aliam, cujus radices sint $\alpha^2, \beta^2, \gamma^2, \delta^2, \&c.$ & resultabit

$v^n + 2q - p^2 v^{n-1} + 2s - 2pr + q^2 v^{n-2} + \&c. = 0;$ quot autem resultantis æquationis signa continuo progrediantur de $+$ in $+$, vel $-$ in $-$; tot vel plures erunt impossibiles datæ æquationis radices.

Sint

Sint enim radices $\alpha, \beta, \gamma, \delta, \&c.$ datae æquationis possibiles, & affirmativæ erunt resultantis æquationis $\alpha^2, \beta^2, \gamma^2, \delta^2, \&c.$ radices; & consequentur tot vel plures erunt mutationes signorum, quot sunt radices possibiles.

Cor. Hæc regula invenit impossibiles radices, cum inveniant nullas prædictæ regulæ, & vice versâ prædictæ regulæ inveniant impossibiles radices, cum nullæ ab hac regulâ detegantur.

Et sic transformetur data æquatio in aliam, cujus radices sint tales functiones datae æquationis radicum, quales sint necessario affirmativæ, cum radix datae æquationis sit possibilis; & numerus resultantis æquationis signorum progressuum continuorum de $+$ in $+$, & $-$ in $-$, erit æqualis vel minor impossibilium radicum numero.

P R O B. XI.

Data æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$ invenire, utrum ullas habet impossibiles radices, necne.

Sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ radices datae æquationis, per Problema quintum transformetur data æquatio in alteram, cujus radices (v) sint $\alpha^2 + \beta^2 - 2\alpha\beta, \alpha^2 + \gamma^2 - 2\alpha\gamma, \alpha^2 + \delta^2 - 2\alpha\delta, \beta^2 + \gamma^2 - 2\beta\gamma, \beta^2 + \delta^2 - 2\beta\delta, \&c.$ æquatio resultans est

$$v^{n \times \frac{n-1}{2}} - n-1 p^2 - 2nq v^{n \times \frac{n-1}{2}-1} + n-1 \times \frac{n-2}{2} p^4 - 2n \times n - 2p^2 q + 2 \times n - 3pr$$

$+ 2ns + 2n + 3 \times n - 2 \times q^2 v^{n \times \frac{n-2}{2}-2} - \&c. = 0$, si mutantur hujusce æquationis resultantis signa alternatim de $+$ in $-$, & $-$ in $+$; nullam radicem impossibilem habet data æquatio: sin hujusce æquationis signa haud continuo mutantur de $+$ in $-$, & $-$ in $+$; impossibiles radices habet data æquatio.

Sint enim omnes datae æquationis radices ($\alpha, \beta, \gamma, \delta, \&c.$) possibiles, & resultantis æquationis radices ($\alpha - \beta, \alpha - \gamma, \alpha - \delta, \beta - \gamma, \&c.$) erunt affirmativæ, & æquationis signa alternatim de $+$ in $-$, & $-$ in $+$ mutabuntur: sint vero duæ datae æquationis radices impossibiles, viz. $a + \sqrt{-b^2}$ & $a - \sqrt{-b^2}$; & una radix resultantis æquationis

nis

tionis erunt impossibiles, prout ultimus solummodo datæ æquationis terminus nihilo sit æqualis, necne.

Paulo facilius reddi potest hæc solutio. Sit ultimus datæ æquationis terminus affirmativa, & vel q affirmativa, vel $q^2 - 4s$ negativa quantitas, & quatuor datæ æquationis radices erunt impossibiles; sin aliter omnes ejus radices erunt possibiles.

Ex. 4. Sit æquatio $x^5 + qx^3 - rx^2 + sx - t = 0$, cujus radices sint $\alpha, \beta, \gamma, \delta, \varepsilon$: æquatio, cujus radices sint $(\alpha - \beta^2, \alpha - \gamma^2, \beta - \gamma^2, \alpha - \delta^2, \beta - \delta^2, \&c.)$ erit $w^{10} + 10qw^9 + 39q^2w^8 + 10s \times w^8 + 80q^3w^7 + 50qs + 25r^2 \times w^7 + 95q^4w^6 + 124q^2s + 95s^2 + 92qr^2 + 200rt \times w^6 + 66q^5w^5 - 360qs^2 + 196q^3s + 118q^2r^2 + 260r^2s + 625t^2 + 400qrt \times w^5 + 25q^6w^4 - 40s^3 - 53r^4 + 52q^3r^2 - 522q^2s^2 + 194q^4s + 708qr^2s + 240q^2rt + 1750qt^2 - 950srt \times w^4 + 4q^7 + 106q^5s - 80qs^3 - 308q^3s^2 - 102qr^4 - 7q^4r^2 + 570r^2s^2 + 612q^2r^2s + 700r^3t - 3750t^2s + 2500t^2q^2 + 80rtq^3 - 2150qrst \times w^3 + 400s^4 - 360q^4s^2 - 15q^4s^2 + 24q^6s - 8q^5r^2 - 45q^2r^4 - 270r^4s + 140r^2sq^3 + 960r^2s^2q + 1875t^2r^2 + 1000trs^2 - 5000t^2qs + 1750t^2q^3 + 40trq^4 + 600tr^3q - 1650trs^2q \times w^2 + 36q^5s^2 - 224q^3s^3 + 320qs^4 + 4q^3r^4 + 27r^6 - 40r^2s^3 + 434r^2q^2s^2 - 24r^2sq^4 - 198r^4qs + 5000t^2s^2 - 450tr^3s - 6250t^3r + 675t^2q^4 - 3750t^2q^2s + 3000t^2r^2q + 60tr^3q^2 + 200trs^2q - 330trq^3s \times w + 3125t^4 - 3750qrt^3 + 2000s^5q + 2250r^2s - 900sq^3 + 825r^2q^2 + 108q^5 \times t^2 - 1600s^3r - 560rq^2s^2 - 16r^3q^3 + 630r^3qs + 72rsq^4 - 108r^5 \times t + 256s^5 - 128q^2s^4 + 144r^2qs^3 + 16q^4s^3 - 27r^4s^2 - 4r^2q^3s^2 = 0$: si continuo mutantur hujusce æquationis signa de + in -, & - in +; nullas impossibiles radices habet data æquatio.

Si signa terminorum æquationis haud continuo mutantur de + in -, & - in +; duæ, vel quatuor datæ æquationis radices erunt impossibiles, prout ultimus ejus terminus sit negativa, vel affirmativa quantitas.

Si ultimus terminus nihilo sit æqualis, & signa terminorum æquationis haud continuo mutantur de + in -, & - in +; tum vel quatuor, vel duæ radices datæ æquationis erunt impossibiles, prout duo, & non plures ultimi datæ æquationis termini nihilo sint æquales, necne.

Cor.

Cor. Eodem prorsus modo cum justâ terminorum comparatione inveniri potest numerus impossibilium radicum in quâcunque æquatione contentarum.

Cor. 2. Sit data æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, transformetur hæc æquatio in alteram, cujus radix est $x - a$, & data & resultans æquatio transformentur in æquationem methodo hoc problemate contentâ, & ex utrâque transformatione eadem resultabit æquatio.

Cor. 3. Si ultimus resultantis æquationis terminus nihilo sit æqualis, tum duæ radices datæ æquationis erunt inter se æquales; si duo ultimi resultantis æquationis termini nihilo sint æquales, tum duæ radices datæ æquationis bis fiunt æquales; si tres vel quatuor ultimi resultantis æquationis termini nihilo sint æquales, tum duæ radices ter vel quater fiunt inter se æquales; & sic deinceps.

P R O B. XII.

Invenire numerum impossibilium radicum in datâ æquatione contentarum e diversâ transformatione.

1. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$ supponatur $ax^n - \overline{a+b} px^{n-1} + \overline{a+2b} qx^{n-2} - \overline{a+3b} rx^{n-3} + \&c. = v$ cujus radix fit v , & resultabit

$$v^n - Av^{n-1} + Bv^{n-2} - Cv^{n-3} - \dots - Pv^3 + Qv^2 + S = 0.$$

Cum duæ datæ æquationis radices sint inter se æquales S erit nihilo æqualis, in omni autem casu penultimus resultantis æquationis terminus nihilo erit æqualis; cum vero duæ radices bis fiant æquales, tum quatuor ultimi resultantis æquationis termini ($S, *, Q, P,$) nihilo erunt æquales; cum vero duæ radices ter vel quater fiant æquales, tum sex vel octo ultimi resultantis æquationis termini nihilo erunt æquales, & sic deinceps; i. e. tot ultimi resultantis æquationis termini nihilo erunt æquales, quot æquales inter se inveniuntur radices datæ æquationis: ergo ex ultimis resultantis æquationis terminis dici potest, utrum duæ, sex, decem, quatuordecim, &c. vel nullæ, quatuor, octo, duodecim, sedecim, &c. radices datæ æquationis inveniuntur

H

im-

impossibiles, & sic de reliquis; ergo e justâ horum terminorum comparisonē inter se inveniri possit numerus radicum impossibilium, quas habet data æquatio.

Ex. 1. Sit data æquatio $x^2 + q = 0$, & supponatur $2x = v$, & æquatio resultans erit $v^2 + 4q = 0$, duæ vero radices per problema erunt impossibiles, necne; prout ultimus resultantis æquationis terminus ($4q$) sit affirmativa vel negativa quantitas: & sic de reliquis exemplis.

Ex. 2. Sit data æquatio $x^3 + qx - r = 0$, supponatur $3x^2 + q = v$, & erit $v^3 + 3qv^2 - 4q^3 - 27r^2 = 0$.

Ex. 3. Sit data æquatio $x^4 + qx^2 - rx + s = 0$, & supponatur $4x^3 + 2qx - r = v$, & æquatio resultans erit $v^4 - 8rv^3 + 4q^3 - 16qs + 18r^2v^2 + 256s^3 - 128s^2q^2 + 16sq^4 + 144r^2sq - 4r^2q^3 - 27r^4 = 0$.

2. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, deducatur æquatio, cujus radices sint semper limites inter radices datæ æquationis, quæ sit

$az^n - \overline{a+b}pz^{n-1} + \overline{a+2b}qz^{n-2} - \overline{a+3b}rz^{n-3} + \&c. = 0$, supponatur $z^n - pz^{n-1} + qz^{n-2} - rz^{n-3} + sz^{n-4} - \&c. = v$; & per Problema quintum inveniatur æquatio cujus radix sit v , & resultat

$v^n - Av^{n-1} + Bv^{n-2} - Cv^{n-3} + \&c. + Qv^2 + Rv + S = 0$, cum duæ radices fiant æquales, S erit nihilo æqualis; cum vero duæ radices bis fiant æquales, S & R evadent nihil; cum ter fiant æquales, S , R & Q fient nihilo æquales, & sic deinceps: utrum vero 0, 4, 8, 12, 16, &c. vel 2, 6, 10, 14, 18, &c. radices fient impossibiles, dici potest ex ultimo resultantis æquationis termino; & sic de reliquis.

Ex. 1. Sit data æquatio $x^2 - px + q = 0$, & æquatio ad limites erit $2z - p = 0$, supponatur $z^2 - pz + q = v$, & resultabit $v + \frac{1}{4}p^2 - q = 0$; & exinde duæ radices erunt impossibiles, necne, prout $\frac{1}{4}p^2 - q$ sit negativa vel affirmativa quantitas.

Ex. 2. Sit $x^3 + qx - r = 0$, & æquatio ad limites erit $3z^2 + q = 0$; supponatur $z^3 + qz - r = v$; & resultabit $v^3 + 2rv + r^2 + \frac{4q^3}{27} = 0$;

&c

& duæ radices erunt possibiles, necne; prout $r^2 + \frac{4q^3}{27}$ sit negativa vel affirmativa quantitas.

Ex. 3. Sit $x^4 + qx^2 - rx + s = 0$, & æquatio ad limites erit $4x^3 + 2qx - r = 0$, supponatur $x^4 + qx^2 - rx + s = v$, & resultabit æquatio $v^3 + \frac{q^3}{2} - 3s \left| v^2 + 3s^2 - q^2s + \frac{q^4}{16} + \frac{9r^2q}{16} \right| v - s^3 + \frac{s^2q}{2} - \frac{sq^4}{16} - \frac{9r^2sq}{16} + \frac{r^2q^3}{64} + \frac{27r^4}{256} = 0$.

Si ultimus hujusce æquationis terminus sit affirmativa quantitas, tum duæ solummodo radices erunt impossibiles.

Et sic de pluribus exemplis.

Cor. 1. Transformetur data æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$ in alteram, cujus radices sint majores vel minores quam radices datæ æquationis per quantitatem (a), & resultat

$$\begin{aligned} z^n + na \quad z^{n-1} + n \times \frac{n-1}{2} a^2 \quad z^{n-2} + n \times \frac{n-1}{2} \times \frac{n-2}{3} a^3 \quad z^{n-3} + \&c. = 0 \\ -p \quad - \frac{n-1}{2} \times pa \quad - \frac{n-1}{2} \times \frac{n-2}{3} pa^2 \\ + \quad q \quad + \frac{n-2}{3} \times \frac{qa}{r} \end{aligned}$$

& æquatio, cujus radices sint limites inter resultanti æquationis radices, sit

$$\begin{aligned} nz^{n-1} + n-1 \times na \quad z^{n-2} + n-2 \times n \times \frac{n-1}{2} a \quad z^{n-3} + n-3 \times n \times \frac{n-1}{2} \times \frac{n-2}{3} a^2 \quad z^{n-4} + \&c. = 0 \\ - \frac{n-1}{2} p \quad - \frac{n-2}{2} \times \frac{n-1}{3} pa \quad - \frac{n-3}{2} \times \frac{n-1}{3} \times \frac{n-2}{3} pa^2 \\ + \quad \frac{n-2}{3} q \quad + \frac{n-3}{3} \times \frac{n-2}{3} \frac{qa}{r} \end{aligned}$$

supponatur $z^n + na \quad z^{n-1} + n \times \frac{n-1}{2} a^2 \quad z^{n-2} + \&c. = w$

$$\begin{aligned} -p \quad - \frac{n-1}{2} pa \\ + \quad q \end{aligned}$$

& æquatio, cujus radix fit w , eadem erit, quicunque sit valor quantitatis (a).

Eadem accommodari possit methodus ad omnia duorum præcedentium casuum exempla.

Cor. 2. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, fingetur eodem modo, quo prius docetur

$$z^{n-1} - \frac{n-1}{n} pz^{n-2} + \frac{n-2}{n} qz^{n-3} - \&c. = 0$$

& $z^n - pz^{n-1} + qz^{n-2} - rz^{n-3} + \&c. = v$ transformentur hæ duæ æquationes in unam, ita ut exterminetur incognita quantitas (z), & æquatio resultans sit

$$v^{n-1} - Pv^{n-2} + Qv^{n-3} - Rv^{n-4} + Sv^{n-5} - \&c. = 0.$$

Sub primo hujusce æquationis termino colloca signum $+$, sub quolibet reliquorum terminorum scribe signum $-$ vel $+$, prout ejus signum est idem vel diversum a signo termini præcedentis, his denique adjiciendum est signum $+$ vel $-$, prout (n) est par vel impar numerus; & tot vel plures erunt radices impossibiles, quot sunt in subscriptorum signorum serie continui progressus de $+$ in $+$, & $-$ in $-$.

Ex. Sit æquatio $x^3 - 9x + 20 = 0$, fingetur $z^2 - 3 = 0$, & $z^3 - 9z + 20 = v$, unde æquatio resultans erit $v^2 - 40v + 292 = 0$

+ + + - sub primo

hujusce æquationis termino (v^2) colloco signum $+$; dein quoniam signum $-$ secundi termini ($-40v$) diversum est a signo $+$ primi termini, sub termino ($-40v$) colloco signum $+$; & eâ ratione sub tertio termino colloco signum $+$, his denique adjeci signum $-$, quoniam n est impar numerus; subscriptorum signorum, quæ in hac serie $+$ $+$ $+$ $-$ sint, una solummodo est mutatio de $+$ in $-$, & consequenter una solummodo erit possibilis radix.

E duobus casibus hujus problematis infinitæ confimiles sequuntur regulæ.

Cor. 3. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c.$ multiplicetur hæc

hæc æquatio in duas arithmeticas series successive

$$a, a+b, a+2b, a+3b, a+4b, \&c. \quad \&$$

$$A, A+B, A+2B, A+3B, A+4B, \&c.$$

& supponantur quantitates resultantes

$$az^n - \overline{a+b} pz^{n-1} + \overline{a+2b} qz^{n-2} - \overline{a+3b} rz^{n-3} + \&c. = \sigma$$

$$\& \quad Az^n - \overline{A+B} pz^{n-1} + \overline{A+2B} qz^{n-2} - \overline{A+3B} rz^{n-3} + \&c. = v,$$

transformentur hæ duæ æquationes in unam ita ut exterminetur incognita quantitates (z), & resultabit æquatio, cujus radix est v , viz.

$v^n - Pv^{n-1} + Qv^{n-2} \dots Sv + T = 0$; multiplicetur data æquatio in duas alias arithmeticas series

$$c, c+d, c+2d, c+3d, \&c.$$

$$C, C+D, C+2D, C+3D, \&c.$$

& supponantur quantitates resultantes

$$cZ^n - \overline{c+d} pZ^{n-1} + \overline{c+2d} qZ^{n-2} - \overline{c+3d} rZ^{n-3} + \&c. = o$$

$$\& \quad CZ^n - \overline{C+D} pZ^{n-1} + \overline{C+2D} qZ^{n-2} - \overline{C+3D} rZ^{n-3} + \&c. = V,$$

transformentur hæ duæ æquationes in unam, ita ut exterminetur incognita quantitas (Z); & resultabit æquatio, cujus radix est V , viz.

$V^n - \alpha V^{n-1} + \beta V^{n-2} - \gamma V^{n-3} \dots \sigma V + \tau = 0$; & erit ultimus prioris resultantis æquationis terminus (T) ad ultimum posterioris

resultantis æquationis terminum $\tau :: \frac{Ba-bA^n}{a^n} : \frac{Dc-dC^n}{c^n}$.

Cor. 4. Sint duæ æquationes

$$nz^{n-1} - \overline{n-1} pz^{n-2} + \overline{n-2} qz^{n-3} - \overline{n-3} rz^{n-4} + \dots \tilde{w} = o$$

$$\& \quad az^n - \overline{a+b} pz^{n-1} + \overline{a+2b} qz^{n-2} - \overline{a+3b} rz^{n-3} + \dots \overline{a+nb} \tilde{w} = v,$$

transformentur hæ duæ æquationes ita ut exterminetur z , & resultat æquatio $v^{n-1} - Pv^{n-2} + Qv^{n-3} \dots Sv + T = o$.

Sint etiam duæ æquationes

$$nz^n - \overline{n-1} pz^{n-1} + \overline{n-2} qz^{n-2} - \overline{n-3} rz^{n-3} + \&c. = o,$$

$$\& \quad az^n - \overline{a+b} pz^{n-1} + \overline{a+2b} qz^{n-2} - \overline{a+3b} rz^{n-3} + \&c. = P$$

& transformentur hæ duæ æquationes in unam, ita ut exterminetur z ,

&

Quantitas, quæ nihilo fit æqualis, cum duæ radices fiunt æquales, & quæ divisores haud admittat, vel evadet negativa, cum 2, 6, 10, 14, 18, &c. radices fiant impossibiles; & affirmativa, cum 0, 4, 8, 12, 16, &c. fiant impossibiles: vel affirmativa, cum 2, 6, 10, 14, 18, &c. radices fiant impossibiles; & negativa, cum 0, 4, 8, 12, 16, &c. fiant impossibiles.

Et consimilia dicenda sunt de quantitativibus, quæ nihilo fiant æquales; cum duæ radices, bis, ter, quater, &c. fiant æquales.

Quantitates enim primum fiunt æquales, deinde impossibiles; & consequenter omnis regula, quæ invenit, cum radices fiant impossibiles, necessario etiam inveniet, cum fiant æquales; & vice versâ, omnis regulæ ope, quæ generaliter inveniat, cum radices semel, bis, ter, &c. fiant æquales, inveniri potest, cum radices fiant impossibiles.

Ex. Sit $x^n - A = 0$, & $(2n)$ radices fiunt æquales, cum A sit $= 0$; & consequenter duæ radices erunt possibiles, necne, prout A sit affirmativa vel negativa quantitas.

Ex. 2. Sit $x^n - Ax^m - B = 0$, & sit primo (n) impar numerus, & $\frac{m-n}{n-m} A^n - \frac{n^m}{m^m} B^{n-m}$ affirmativa quantitas, & tres radices erunt possibiles; sin aliter, una solummodo invenitur possibilis radix: sit (n) par numerus, & $-B$ negativa quantitas, & duæ solummodo inveniuntur possibiles radices: sit (n) par numerus, m vero impar, & $-B$ affirmativa quantitas, & duæ radices erunt possibiles, si $\frac{m-n}{n-m} A^n - \frac{n^m}{m^m} B^{n-m}$ sit negativa quantitas, sin aliter nullæ: sint n & m pares numeri, & $-B$ & $-A$ affirmativæ quantitates, & nullam habet possibilem radicem data æquatio: sit $-B$ affirmativa, & $-A$ negativa quantitas, & sit $\frac{m-n}{n-m} A^n - \frac{n^m}{m^m} B^{n-m}$ affirmativa, & quatuor possibiles radices habet data æquatio; sin aliter nullas.

THEOREM. 6.

1. Sit cubica æquatio $x^3 + px^2 + qx + r = 0$, cujus duæ radices sint impossibiles, & tertia erit negativa vel affirmativa quantitas, prout (r) sit affirmativa vel negativa quantitas.

2. Sit æquatio biquadratica $x^4 + px^3 + qx^2 + rx + s = 0$, cujus duæ radices sint impossibiles & duæ possibiles: sit s negativa quantitas, & altera possibilis radix erit negativa, altera vero affirmativa: sit s affirmativa quantitas, & termini continuo mutantur de $+$ in $-$, & $-$ in $+$; tum duæ possibiles radices erunt affirmativæ; si omnes termini sint affirmativi, tum duæ possibiles radices erunt negativæ: si termini haud continuo mutantur de $+$ in $-$, & $-$ in $+$; vel sint omnes affirmativæ; tum duæ possibiles radices erunt negativæ vel affirmativæ, prout $p^3 - 4pq + 8r$ sit affirmativa vel negativa quantitas.

Cor. Sit æquatio $x^4 + qx^2 + rx + s = 0$, cujus duæ sint possibiles radices; & sit s affirmativa quantitas, & duæ possibiles radices erunt negativæ vel affirmativæ, prout (r) sit affirmativa vel negativa quantitas.

3. Sit $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + sx^{n-4} \dots Rx + S = 0$ cujus ultimus terminus S sit negativa quantitas, & impar erit numerus affirmatarum radicum in datâ æquatione contentarum; sit S affirmativa quantitas, & par erit numerus affirmatarum radicum. Si termini datæ æquationis continuo mutantur de $+$ in $-$, & $-$ in $+$; tum omnes possibiles radices erunt affirmativæ; si termini sint omnes affirmativi, tum omnes possibiles radices erunt negativæ.

Ex. 1. Sit $x^{2n+1} + A = 0$, & possibilis radix vel erit affirmativa vel negativa, prout A sit negativa vel affirmativa quantitas: sit $x^{2n} - A^2 = 0$ & una possibilis radix erit affirmativa, altera vero negativa quantitas.

Ex. 2. Sit $x^{2n} + Ax^m + B = 0$, & si duæ solummodo radices sint possibiles, & B negativa quantitas, una radix erit affirmativa, altera vero negativa quantitas; sit B affirmativa quantitas, & duæ possibiles radi-

ces erunt negativæ vel affirmativæ, prout A sit affirmativa vel negativa quantitas.

Sint quatuor possibiles, & duæ erunt affirmativæ, duæ vero negativæ radices.

Sit $x^{2n+1} + Ax^n + B = 0$, & si una radix solummodo sit possibilis, ea erit affirmativa vel negativa quantitas, prout B sit negativa vel affirmativa quantitas. Si vero tres sint possibiles, & B affirmativa quantitas; & tres possibiles radices erunt una negativa, duæ vero reliquæ affirmativæ.

Sit B negativa quantitas, & tres possibiles radices erunt una affirmativa, duæ vero reliquæ negativæ.

4. Dato impossibilium radicum numero, invenire summam negativarum vel positivarum radicum in datâ æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$ contentarum.

Sint quadratici datæ æquationis divisores $x^2 - 2ax + a^2 \pm A^2 = 0$, $x^2 - 2bx + b^2 \pm B^2 = 0$, $x^2 - 2cx + c^2 \pm C^2 = 0$, $x^2 - 2dx + d^2 \pm D^2 = 0$, $x^2 - 2ex + e^2 \pm E^2 = 0$, &c.; inveniantur quantitates, quæ nihilo fiant æquales, cum A^2 vel B^2 , C^2 , D^2 , E^2 , &c. nihilo fiant æquales, & quæ nullum recipiant divisorem, & e justâ comparatione quantitatum inter se resultantium inveniri possit numerus affirmativarum & negativarum radicum.

Hinc constat, quod in æquationibus superiorum dimensionum calculus requiratur maxime operosus.

P R O B. XIV.

Concessâ methodo detegendi numerum impossibilium datæ æquationis radicum, in quibus continetur una solummodo incognita quantitas: Concessâ etiam methodo inveniendi, utrum quædam sint possibiles radices incognitarum quantitatum datarum æquationum, quæ inter datos limites constituantur: Et data æquatione, quæ plures habet quam unam incognitam quantitatem; invenire, utrum hæc incognitæ quantitates possibiles admittant valores, necne.

Assumantur omnes præter unam incognitæ quantitates tanquam

datae; & inveniatur e concessis principiis, quando omnes valores huius incognitae quantitatis fiant impossibiles, & resultabunt quadam aequationes; e quibus exterminetur praedicta incognita quantitas.

Et sic de reliquis incognitis quantatibus exterminandis, usque donec resultant aequationes, in quibus unica solummodo continetur incognita quantitas; & ex hisce aequationibus inveniatur, utrum incognita quantitas possibiles habere potest valores, necne: si vero possibiles valores admittant aequationes praedictae; investigandum est, an possibiles valores possint esse inter eosdem limites contenti.

Ex. 1. Sit $y^2 + 2z + 8xy + 2z^2 + 10z + 20 = 0$; invenire num incognitae quantitates z & y possibiles correspondentes valores admittant, necne.

Assumatur z tanquam data quantitas, & notum est quantitatem (y) nullos admittere possibiles valores; si modo $z+4 = 2z^2 - 10z - 20$, i.e. $-z^2 - 2z - 4$ semper sit negativa, vel $z^2 + 2z + 4$ semper sit affirmativa quantitas. Radices autem aequationis $z^2 + 2z + 4 = 0$ sunt impossibiles, ergo $z^2 + 2z + 4$ semper erit affirmativa quantitas, & nullos possibiles correspondentes valores admittant incognitae quantitates z & y .

Ex. 2. Sit aequatio $y^2 + 4 + 2x + 4xy + 2x^2 + 6z^2 + 5xz + 6 = 0$, si modo omnes valores quantitatis (y) sint impossibiles, tum $x^2 + z - 4 \mid x + 2z^2 - 8z + 2$ semper erit affirmativa quantitas; ut semper fiat affirmativa quantitas, necesse est, ut radices aequationis $x^2 + z - 4 \mid x + 2z^2 - 8z + 2 = 0$ sint impossibiles, i.e. ut $\frac{z-4}{4} - 2z^2 + 8z - 2$ semper sit negativa quantitas, i.e. ut radices aequationis $7z^2 - 24z - 8 = 0$ sint impossibiles; sed radices huius aequationis haud erunt impossibiles; erunt enim $\frac{12 \pm 10\sqrt{2}}{7}$; & exinde sequitur, si modo radices quantitatis (x) sint possibiles, tum radices quantitatis (z) inter $\frac{12 + 10\sqrt{2}}{7}$ & $\frac{12 - 10\sqrt{2}}{7}$ consistent: assumatur igitur

quæ-

quæcunque quantitas inter $\frac{12+10\sqrt{2}}{7}$ & $\frac{12-10\sqrt{2}}{7}$ pro z , quâ pro suo valore in æquatione $x^2 + \frac{z-4}{2}x + 2z^2 - 8z + 2 = 0$ substituta, & inveniantur duæ radices quantitatis (x) exinde resultantes, assumantur quæcunque quantitas inter has duas radices, quâ pro x in primâ æquatione $y^2 + 4 + 2x + 4z \times y + 2x^2 + 6z^2 + 5xz + 6 = 0$, & ex eâ resultabunt duæ radices quantitatis (y) possibiles.

Eodem modo invenietur quantitas x inter duas quantitates $2\sqrt{19} - 4$ & $-4 - 2\sqrt{19}$ posita, si modo z sit possibilis: & sic de quantitatibus x & y , z & y , & exinde x & y & z .

Cor. Datâ methodo inveniendi, quot impossibiles radices habeat data æquatio, & datâ methodo transformandi datam æquationem in alteram vel alias; ex his principiis inveniri possit, quot possint esse impossibiles radices in resultanti æquatione & inter quos limites contineatur quicumque impossibilium radicum numerus.

Et sic de inventione numeri affirmatarum & negatarum radicum, quæ in datâ æquatione contineri potest.

P R O B. XV.

Cognito datæ æquationis impossibilium etiamque affirmatarum & negatarum radicum numero; & data methodo inveniendi, inter quos limites consistant possibiles valores cujuscunque functionis datæ æquationis radicum: & transformetur data æquatio in alteram, cujus radices sint data functio datæ æquationis radicum; invenire quot impossibiles radices habet resultans æquatio.

Hoc problema generalem resolutionem a problematis præcedentis corollario recipiat; sed in multis casibus multo facilius juxta subsequentem methodum solvi possit.

Primo e datâ functione perscrutandum est, cujus generis sint radices datæ æquationis, quæ dabunt impossibiles radices in resultanti æquatione, vel cujus generis sint radices datæ æquationis, quæ dabunt possibiles radices in resultanti æquatione.

Hæc e terminis irrationalibus diversis, & rationalibus sejunctim æstimatis semper constabunt.

Consimili autem methodo deduci possit numerus negativarum & affirmatarum radicum.

Hæc autem per exempla facilius doceri possint.

Ex. 1. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, cujus radices sint $\alpha, \beta, \gamma, \delta, \&c.$; & transformetur hæc æquatio in aliam, cujus radices sint $\alpha^2, \beta^2, \gamma^2, \delta^2, \&c.$ Et primo omnis possibilis radix in datâ æquatione alteram habet in resultanti æquatione sibi correspondentem: 2^{do} omnis impossibilis radix in datâ æquatione habet impossibilem radicem in resultanti sibi correspondentem, ni radix datæ æquationis sit hujusce formulæ $\sqrt{-A^2}$; ergo per caput subsequens inveniendum est, quot radices hujusce formulæ $\sqrt{-A^2}$ habet data æquatio, qui numerus sit (m); & in æquatione resultante minor impossibilium radicum invenitur numerus, quam in datâ æquatione per numerum (m): numerus autem negativarum radicum erit m ; tot autem habet affirmativas radices resultans æquatio, quot possibiles habet data æquatio.

Ex. 2. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, cujus radices sint $\alpha, \beta, \gamma, \delta, \&c.$ quarum sint $2r$ impossibiles, & $n - 2r$ possibiles; & transformetur hæc æquatio in alteram, cujus radices sint $\alpha - \beta, \alpha - \gamma, \beta - \gamma, \alpha - \delta, \beta - \delta, \gamma - \delta, \&c.$

Investigandum est, cujus generis sint radices datæ æquationis, quæ habeant impossibiles radices in resultanti æquatione correspondentes.

Sint 1^{mo} α & β rationales vel negativæ vel affirmativæ quantitates, & $\alpha - \beta$, quæ est radix resultantis æquationis, erit rationalis & affirmativa quantitas.

2^{do}. Sint duæ correspondentes datæ æquationis radices impossibiles $a + b\sqrt{-1}$ & $a - b\sqrt{-1}$; & $a + b\sqrt{-1} - a + b\sqrt{-1} = 2b\sqrt{-1} = -4b^2$, quæ erit radix resultantis æquationis, erit possibilis & negativa quantitas.

Sint ergo nullæ radices inter se æquales, nec ullæ radices hujusce generis

generis $a + \sqrt{-b^2}$ & $a + \sqrt{-c^2}$, vel $a + \sqrt{-b^2}$ & $c + \sqrt{-b^2}$: & radices resultantis æquationis erunt $\frac{n-2r}{2} \times \frac{n-2r-1}{2}$ (affirmativæ), $+ r$ negativæ $+ 2rn - 2r^2 - 2r$ impossibiles $= n \times \frac{n-1}{2}$.

Si vero ullæ sint radices datæ æquationis vel inter se æquales, vel harum formularum $a + \sqrt{-b^2}$ & $a + \sqrt{-c^2}$; vel $a + \sqrt{-b^2}$ & $c + \sqrt{-b^2}$; inveniri possunt methodo in subsequenti capite traditâ radices harum formularum: quarum omnium cognito numero, exinde facile inveniri possit numerus affirmativarum, negativarum & impossibilium radicum in æquatione resultanti.

Ex. 3. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} \&c. = 0$, cujus radices sint $\alpha, \beta, \gamma, \delta, \&c.$ quarum $2r$ sint impossibiles, cæteræ vero possibiles.

Transformetur hæc æquatio in alteram, cujus radices sint $nx^{n-1} - \frac{n-1}{1} px^{n-2} + \frac{n-2}{2} qx^{n-3} - \frac{n-3}{3} rx^{n-4} + \frac{n-4}{4} sx^{n-5} \&c. = v$ si nullæ sint radices duarum æquationum

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0 \&$$

$$nx^{n-1} - \frac{n-1}{1} px^{n-2} + \frac{n-2}{2} qx^{n-3} - \frac{n-3}{3} rx^{n-4} + \&c. = a:$$

inter se æquales, & nullam radicem impossibilem ($a + \sqrt{-b^2}$) hujusmodi admittat data æquatio, ita ut $n \times \frac{n-1}{1} a^{n-2} - n \times \frac{n-1}{1} \times \frac{n-2}{2} \times$

$$\frac{n-3}{3} a^{n-4} b^2 + n \times \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \frac{n-4}{4} \times \frac{n-5}{5} a^{n-6} b^4 - \&c. =$$

$$\frac{n-1}{1} \times \frac{n-2}{2} p a^{n-3} + \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \frac{n-4}{4} p a^{n-5} b^2 - \frac{n-1}{1} \times \frac{n-2}{2}$$

$$\times \frac{n-3}{3} \times \frac{n-4}{4} \times \frac{n-5}{5} \times \frac{n-6}{6} p a^{n-7} b^4 + \&c. + \frac{n-2}{2} \times \frac{n-3}{3} q a^{n-4} \&c. = 0:$$

tum æquatio resultans habebit radices impossibiles ($2r$), & si n sit par numerus, $\frac{n-2r}{2}$ & affirmativas & negativas radices; si vero (n)

fit

fit impar numerus, $\frac{n-2r-1}{2}$ negativas, & $\frac{n-2r+1}{2}$ affirmativas; habebit resultans æquatio.

Utrum vero ullæ prædictarum æquationum radices sint inter se æquales, & exinde quædam radices resultantis æquationis nihilo fiant æquales, vel ulla quantitas prædictæ formulæ nihilo sit æqualis, & exinde quædam sunt impossibiles radices datæ æquationis, quæ possibilibus radicibus æquationis resultantis respondent, e subsequenti capite colligi potest.

Et vice versâ dato impossibilium radicum in resultanti æquatione contentarum numero, inveniri possit impossibilem datæ æquationis radicum numerus.

Eadem erit methodus ratiocinandi in hoc ac in priori casu.

E. G. Sit data æquatio $x^3 + qx - r = 0$, cujus radices sint $\alpha, \beta, -\alpha - \beta$; & data æquatio erit $x^3 - \alpha^2 + \alpha\beta + \beta^2 x + \alpha\beta x + \alpha + \beta = 0$: transformetur hæc æquatio in alteram, cujus radix sit z ; & sit relatio inter x & z per æquationem $z^3 - xz = \frac{q}{3} = -\frac{\alpha^2 + \alpha\beta + \beta^2}{3}$ expressa,

& resultat æquatio $z^6 - rz^3 = \frac{q^3}{27}$. Sint primum omnes tres radices datæ æquationis possibiles, & substituantur pro (x) in æquatione $z^3 - xz = -\frac{\alpha^2 + \alpha\beta + \beta^2}{3}$ successive α, β & $-\alpha - \beta$; & resultant tres

æquationes $z^3 - \alpha z = -\frac{\alpha^2 + \alpha\beta + \beta^2}{3}$, $z^3 - \beta z = -\frac{\alpha^2 + \alpha\beta + \beta^2}{3}$, $z^3 + \alpha + \beta z = -\frac{\alpha^2 + \alpha\beta + \beta^2}{3}$, & exinde $z = \frac{\alpha}{2} + \frac{\alpha + 2\beta \times \sqrt{-3}}{6}$ & $z = \frac{\beta}{2} + \frac{\beta + 2\alpha \times \sqrt{-3}}{6}$, & $z = -\frac{\alpha + \beta}{2} + \frac{\alpha - \beta \sqrt{-3}}{6}$; ergo si ra-

dices datæ æquationis sint omnes possibiles, radices resultantis æquationis erunt omnes impossibiles: sint duæ radices (x) datæ æquationis impossibiles $\alpha + \sqrt{-A^2}$ & $\alpha - \sqrt{-A^2}$; tertia autem erit -2α ,
&

& data æquatio $x^3 - 3a^2 - A^2x + 2a^3 + 2A^2a = 0$; supponatur $z^2 -$
 $zx = \frac{q}{3} = -a^2 + \frac{A^2}{3}$; substituantur pro (x) ejus tres respectivi va-
 lores successive, & resultabunt tres quadraticæ æquationes, quarum
 radix est z , viz. $z^2 - a + \sqrt{-A^2} z = -a^2 + \frac{A^2}{3}$, & $z^2 - a - \sqrt{-A^2}$
 $z = -a^2 + \frac{A^2}{3}$, & $z^2 + 2ax = -a^2 + \frac{A^2}{3}$: duarum priorum æ-
 quationum omnes radices erunt impossibiles, ultimæ autem æqua-
 tionis radices (z) erunt $z = \frac{A}{\sqrt{3}} - a$: ergo quatuor impossibiles
 & duas possibiles radices habet resultans æquatio $z^6 - rz^3 = \frac{q^3}{27}$, si
 modo duas impossibiles habeat data æquatio $x^3 + qx - r = 0$: sin ali-
 ter, nullas: & vice versâ, si modo resultans æquatio $z^6 - rz^3 = \frac{q^3}{27}$
 duas habeat possibiles radices, duas impossibiles habebit data æqua-
 tio: sin aliter nullas.

Ex. 2. Sit æquatio $x^4 + qx^2 - rx + s = 0$, cujus radices sint $\alpha, \beta,$
 γ, δ : & æquatio, cujus radices sint summæ quarumcunque duarum
 respective, i. e. sint $\alpha + \beta, \alpha + \gamma, \beta + \gamma, \alpha + \delta, \beta + \delta, \gamma + \delta$, erit

$$z^6 + 2qz^4 + \frac{q^2}{4s} \} z^2 - r^2 = 0.$$

Si omnes datæ æquationis radices $(\alpha, \beta, \gamma, \delta)$ sint possibiles, tum
 omnes resultantis æquationis radices erunt possibiles: si vero duæ
 vel quatuor datæ æquationis radices sint impossibiles, tum quatuor
 resultantis æquationis radices erunt impossibiles, & duæ possibiles,
 ni duæ radices impossibiles bis fiant æquales, i. e. æquatio sit $x^4 +$
 $2A^2x^2 + A^4 = 0$, in quo casu omnes radices datæ æquationis erunt
 impossibiles, duæ vero resultantis æquationis erunt impossibiles &
 quatuor nihilo æquales.

Sint enim quatuor radices datæ æquationis impossibiles & respective
 per $a \pm A\sqrt{-1}$, $-a \pm B\sqrt{-1}$ designatæ, & quatuor resultantis æ-
 qua-

æquationis impossibiles radices erunt respectivè $\overline{B+A\sqrt{-1}}$, $-\overline{B+A\sqrt{-1}}$, $\overline{B-A\sqrt{-1}}$, $-\overline{B-A\sqrt{-1}}$; duæ vero possibiles radices erunt $2a$ & $-2a$; & earum quadrata erunt quatuor negativæ quantitates, duæ vero affirmativæ; & quoniam negativæ & affirmativæ resultantis æquationis radices sunt inter se æquales; æquatio $v^3 + 2qv^2 + q^2 - 4s v - r^2 = 0$, cujus radices sint quadrata radicum resultantis æquationis, i.e. $v = x^2$, habebit omnes ejus radices possibiles, duas vero negativas & unam affirmativam, ni $a=0$ & $A=B$; in quo casu data æquatio erit $x^4 + 2A^2x^2 + A^4 = 0$; & resultantis æquationis quatuor radices nihilo erunt æquales, duæ vero reliquæ invenientur impossibiles.

Sint duæ radices datæ æquationis possibiles, duæ vero impossibiles; & per $\alpha, \beta, -\frac{\alpha+\beta}{2} + \sqrt{-A^2}$, $-\frac{\alpha+\beta}{2} - \sqrt{-A^2}$ respectivè designatæ; & erunt quatuor resultantis æquationis radices impossibiles, viz. $\frac{\alpha-\beta}{2} \pm \sqrt{-A^2}$, $\frac{\beta-\alpha}{2} \pm \sqrt{-A^2}$, duæ vero possibiles $\alpha+\beta$ & $-\alpha-\beta$; impossibilium radicum quadrata etiam erunt impossibiles quantitates; ni $\alpha=\beta$, in quo casu duæ radices cubicæ æquationis $v^3 + 2qv^2 + q^2 - 4s v - r^2 = 0$, cujus radices sint quadrata radicum resultantis æquationis, erunt inter se æquales; erunt enim $-A^2$, $-A^2$ & $4a^2$.

Si igitur cubicæ æquationis $v^3 + 2qv^2 + q^2 - 4s v - r^2 = 0$ omnes radices sint possibiles & affirmativæ, omnes radices datæ biquadraticæ etiam erunt possibiles; si omnes radices æquationis cubicæ sint possibiles, duæ vero radices negativæ, & una affirmativa, quatuor radices datæ æquationis erunt impossibiles: si vero duæ radices cubicæ sint impossibiles, tum duæ radices biquadraticæ etiam erunt impossibiles, duæ vero possibiles.

Cor. Datâ generali reductione æquationum, facile hinc constabit methodus inveniendi, quot impossibiles radices habet data æquatio; si modo enim reducantur æquationes, necesse est, ut transformetur æqua-

æquationum quoad formulam minores dimensiones habentium ope, data æquatio: data autem relatio inter radices datæ & transformatæ æquationis semper exprimitur per æquationes quoad formulas earum minores quam datæ æquationis dimensiones habentes, ergo e numero earum impossibilium radicum constat numerus impossibilium radicum in datâ æquatione.

S C H O L.

Modus demonstrandi maxime accommodatus hisce rebus, ubi quæritur, quot radices impossibiles continentur in datâ æquatione; est vel substituere quantitates ipsas pro suis valoribus; vel multiplicare datam æquationem 1^{mo} per $x - a = 0$; deinde per $x^2 - 2ax + a^2 + b^2 = 0$; cum enim prior æquatio unicam radicem habeat, eamque possibilem; posterior autem duas impossibiles; manifestum est eundem futurum esse numerum radicum impossibilium in æquatione, quæ oritur e priori multiplicatione, ac in datâ æquatione: æquatio autem, quæ oritur e secundâ multiplicatione duas habebit radices impossibiles, quæ in datâ æquatione non contineantur.

Ex. Sit data æquatio $x^2 - 2x + 2 = 0$, quæ secundum Newtoni regulam duas habet radices impossibiles; multiplicetur per $x + 2 = 0$, & fit $x^3 - 2x + 4 = 0$, quæ secundum eandem regulam nullas habet radices impossibiles, debet autem habere duas; regula igitur non semper accurata est.

Modus autem demonstrationis, de quo loquor, non huic quæstioni solummodo proprius est, sed commode adhiberi potest, ubicunque quæritur, quot quantitates dati cujuscvis generis in datâ æquatione contineantur.

Ex. I. Sit notissima Cartesii regula, "tot esse radices affirmativas, quot signorum in continuâ serie mutationes de + in - & - in +; si modo nullæ æquationis radices sint impossibiles."

Si enim data quævis æquatio multiplicetur per $x - a = 0$, mutationum signorum numerus augebitur per unitatem, si autem eadem æquatio multiplicetur per $x + a = 0$, eadem erunt signorum mutationes in novâ, ac in datâ æquatione.

K

Sit

Sit enim data æquatio

$$x^n - Ax^{n-1} - Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + Ex^{n-5} + Fx^{n-6} + \&c. = 0$$

quæ ducta in

$$x - a$$

$$\text{dat } x^{n+1} - Ax^n - Bx^{n-1} + Cx^{n-2} + Dx^{n-3} + Ex^{n-4} + \&c. = 0.$$

$$-a \quad +Aa \quad +aB \quad -aC \quad -aD$$

jam vero quod ad signa hujus novæ æquationis attinet, manifestum est,

1. Terminum quemcunque novæ æquationis idem habere signum cum termino datæ æquationis, cui subscribitur, si modo coefficientis istius termini major sit quam coefficientis termini antecedentis multiplicata per (*a*). E. G. Signum quinti termini in datâ æquatione est affirmativum, scil. (+ *D*), signum autem termini quinti in novâ æquatione est (*D* — *aC*), quod etiam affirmativum est, si modo *D* major est quam *aC*.

2. Si vero coefficientis alicujus termini in datâ æquatione minor sit quam coefficientis termini antecedentis multiplicata in (*a*); tum terminus novæ æquationis, qui termino proposito subscribitur, diversum habet signum a termino antecedente in datâ æquatione.

Sit ex. gr. coefficientis termini quinti in datâ æquatione minor quam (*aC*) coefficientis termini antecedentis multiplicata per (*a*), & (*D* — *aC*) coefficientis termini quinti in novâ æquatione est negativa, coefficientis autem termini antecedentis in datâ æquatione est affirmativa + *C*. Ex. G. Sit igitur *B* major quam *Aa*, *C* quam *aB*, *D* autem minor quam *aC*: jam vero quia *D* minor est quam *aC*, erit etiam *E* minor quam *aD*, & *F* quam *aE*, &c. usque ad finem æquationis, cum enim omnes radices datæ æquationis sint possibiles, notum est

$$EC \text{ minorem esse quam } D^2$$

$$\text{est etiam } D \text{ minor quam } aC$$

$$\text{ergo } E \text{ minor est quam } aD.$$

Et duæ æquationes hoc modo scribi possunt

$$x^n - Ax^{n-1} - Bx^{n-2} + Cx^{n-3} + Dx^{n-4} - Ex^{n-5} - Fx^{n-6} + \&c. = 0$$

$$x^{n+1} - A + ax^n - B - Aax^{n-1} + C + aBx^{n-2} - aC - Dx^{n-3} - aD - Ex^{n-4} - aE - Fx^{n-5} + \&c. = 0.$$

Potes

Patet igitur a termino primo ad terminum quartum in utrâque æquatione signa esse eadem. A termino autem quarto in datâ æquatione, & a termino quinto in nova, signa esse semper diversa.

In utroque casu igitur eadem sunt mutationes signorum: est autem in novâ æquatione a termino quarto ad quintum unica mutatio signorum, quæ non est in datâ æquatione.

Eodem modo pateret numerum mutationum signorum nec augeri nec minui multiplicatione per $(x+a)$, unde sequitur tot esse radices affirmativas, quot sunt mutationes signorum de $+$ in $-$, & $-$ in $+$, cæteras vero negativas esse.

Exemplum alterum sit subsequens theorema.

Omnis æquatio tot vel plures habet mutationes ejus coefficientium signorum de $+$ in $-$, & $-$ in $+$, quot sunt ejus affirmativæ radices; & tot vel plures continuos progressus de $+$ in $+$, & $-$ in $-$, quot sunt ejus radices negativæ.

1^{mo}. Sit data æquatio $x^n - Ax^{n-1} - Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + Ex^{n-5} + Fx^{n-6} + Gx^{n-7} + Hx^{n-8} + Ix^{n-9} + Kx^{n-10} + Lx^{n-11} + Mx^{n-12} + Nx^{n-13} \&c. = 0$, & sit A^2 major quam B , B^2 quam AC , C^2 quam DB , sed D^2 minor quam CE ; & tum progressus signorum de Cx^{n-3} ad Dx^{n-4} nec denotat affirmativam, nec negativam radicem, sed impossibilem: sit D^2 minor quam EC , E^2 quam DF , F^2 quam GE , G^2 quam FH , H^2 quam GI , & sic ad terminum Mx^{n-12} ; sed sit M^2 major quam LN ; & progressus signorum de Lx^{n-11} ad Mx^{n-12} nec denotat negativam, nec affirmativam radicem, sed impossibilem; progressui igitur terminorum de Cx^{n-3} ad Dx^{n-4} vel Lx^{n-11} ad Mx^{n-12} & consimilibus nulla habenda est ratio. Hoc posito, ducatur data æquatio

$$x^n - Ax^{n-1} - Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + Ex^{n-5} + Fx^{n-6} + Gx^{n-7} + Hx^{n-8} + Ix^{n-9} \&c. = 0$$

in $x - a$

$$\& \text{ fit } x^{n+1} - Ax^n - Bx^{n-1} + Cx^{n-2} + Dx^{n-3} + Ex^{n-4} + Fx^{n-5} + Gx^{n-6} + Hx^{n-7} + Ix^{n-8} + \&c. = 0$$

$$-a + Aa \quad +Ba \quad -aC \quad -aD \quad -aE \quad -aF \quad -aG \quad -aH$$

Per exemplum præcedens constat theorema verum esse quoad terminos, in quibus haud invenitur prædicta impossibilium radicum nota:

Per notam impossibilium radicum intelligo quadratum coefficientis

alicujus termini minus esse quam coefficientem termini antecedentis multiplicatam in coefficientem termini subsequens.

Sint etiam A, B, C termini, in quibus haud invenitur prædicta impossibilium radicum nota, & D, E, F, G, H, I, K, L termini, in quibus invenitur prædicta nota, & M terminus, in quo haud invenitur prædicta nota; tum signa coefficientium $C, D, E, F, G, H, I, K, L$, vel alternatim continuo mutantur de $+$ in $-$, & $-$ in $+$; vel omnia erunt affirmativa, vel omnia negativa.

Sit A^2 major quam B , B^2 quam AC , C^2 quam BD , sed D^2 minor quam CE , E^2 quam DF , F^2 quam GE , G^2 quam FH , H^2 quam GI , I^2 quam HK , K^2 quam IL , L^2 quam KM , sed M^2 major quam LN : sint etiam D minor quam aC , E quam aD , F quam aE , G quam aF ; sed H major quam aG , & consequenter I major erit quam aH , K quam aI , L quam aK , &c. usque ad finem terminorum prædictam notam impossibilium radicum habentium, i. e. hi termini novæ æquationis eadem habebunt signa cum terminis datæ æquationis, quibus subscibuntur. Per hypothesein enim

GI major est quam H^2

& H quam aG

ergo I major erit quam aH ,

& similiter K major erit quam aI , L quam aK , &c. Et hæ æquationes hoc modo scribi possunt

$$\begin{array}{ccccccc} x^n - Ax^{n-1} - Bx^{n-2} + Cx^{n-3} & + & Dx^{n-4} & + & Ex^{n-5} & + & Fx^{n-6} & + & \&c. \\ x^{n+1} & \dots\dots\dots & -aC - Dx^{n-3} & - & aD - Ex^{n-4} & - & aE - Fx^{n-5} & - & aF - Gx^{n-6} \\ + & H - aGx^{n-7} & + & I - aHx^{n-8} & + & K - aIx^{n-9} & + & L - aKx^{n-10}. \end{array}$$

Patet igitur e priori exemplo a termino primo ad terminum quartum, in quibus haud invenitur prædicta impossibilium radicum nota; vel eundem esse numerum mutationum signorum de $+$ in $-$, & $-$ in $+$, in novâ ac in datâ æquatione; vel eundem esse numerum cum unâ mutatione in novâ, quæ non invenitur in datâ.

A termino quinto Dx^{n-4} ad terminum octavum Gx^{n-7} signa in utrâque æquatione esse semper diversa, & a termino octavo ad terminum tredecimum signa esse semper eadem in novâ ac in datâ æquatione.

A termino quinto Dx^{n-4} ad terminum tredecimum Mx^{n-12} (usque donec haud prædicta sit impossibilium radicum nota) in datâ æquatione signa vel continuo variantur de $+$ in $-$, & $-$ in $+$, vel nullam variationem admittant.

In nova æquatione a termino quinto $aC - Dx^{n-3}$ ad terminum octavum $aF - Gx^{n-6}$, & a termino octavo ad terminum decimum quartum signa vel continuo variantur de $+$ in $-$, & $-$ in $+$, vel continuo eadem erunt: ergo in novâ ac in datâ æquatione idem erit hoc in casu signorum mutationum numerus.

Nulla autem habenda est ratio progressui signorum in datâ æquatione a termino quarto ad quintum, & a termino duodecimo ad terminum tredecimum, hi progressus enim impossibiles radices denotant; & exceptis his prædictis signorum progressibus, qui denotant impossibiles radices, semper invenietur unica mutatio in novâ, quæ non invenitur in datâ æquatione.

Et iisdem exceptis pateret numerum mutationum signorum nec augeri nec minui multiplicatione per $(x+a)$, ergo constat exemplum.

Eodem modo, multiplicatione per $x \pm a$ & $x^2 - 2ax + a^2 + b^2$, demonstrari possit progressus signorum a termino quarto ad quintum, & a termino duodecimo ad terminum tredecimum in datâ æquatione impossibiles designare radices.

CAP. III.

DE REDUCTIONE

ET

RESOLUTIONE ÆQUATIONUM.

PROB. XVI.

DATIS duabus æquationibus $x^n - p x^{n-1} + q x^{n-2} - r x^{n-3} + \&c. = 0$, & $x^m - P x^{m-1} + Q x^{m-2} - R x^{m-3} + \&c. = 0$; prioris æquationis (r) radices sint etiam (r) radices posterioris æquationis.

Reducere priorem æquationem in duas, quarum una habet dimensiones (r) , altera vero $(n-r)$; posteriorem vero in duas alias, quarum una habet dimensiones (r) , & cujus radices eadem erunt ac (r) radices prædictæ æquationis; altera vero $(m-r)$.

Duarum datarum æquationum inveniatur maximus communis divisor, cujus dimensiones erunt (r) ; nihilo æqualis supponatur, & erit una æquatio quæsitæ, quæ utrisque datis æquationibus communis est: dividantur datæ æquationes per hunc communem divisorem, respectivæ quotientes nihilo æquales supponantur, & erunt duæ reliquæ æquationes, quarum dimensiones sunt $n-r$, & $m-r$ respective.

Ex. 1. Sint duæ æquationes $x^5 - 2x^4 + x^3 + 20x^2 - 68x + 48 = 0$ & $x^3 + 2x^2 - 13x + 10 = 0$.

Harum æquationum inveniatur maximus communis divisor $x^2 - 3x + 2$; dividantur datæ æquationes $x^3 + 2x^2 - 13x + 10 = 0$, & $x^5 - 2x^4 + x^3 + 20x^2 - 68x + 48 = 0$ per hunc communem divisorem; & reducitur æquatio $x^3 + 2x^2 - 13x + 10 = 0$ in quadraticam $x^2 - 3x + 2 = 0$, & simplicem $x + 5 = 0$; & æquatio $x^5 - 2x^4 + x^3 + 20x^2 - 68x + 48 = 0$ in cubicam $x^3 + x^2 + 2x + 24 = 0$ & communem quadraticam $x^2 - 3x + 2 = 0$.

Ex. 2. Datâ æquatione $x^n - p x^{n-1} + q x^{n-2} - \&c. = 0$, cujus radices

dices sint $\alpha, \beta, \gamma, \delta, \&c.$ i.e. sit $x^n - px^{n-1} + qx^{n-2} - \&c. = \overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \&c.$; datis etiam quibusdam ex ejus radicibus α, β, γ ; vel data æquatione (m) dimensionum $\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \&c. = 0$, cujus (m) radices sint $\alpha, \beta, \gamma, \&c.$ radices datæ æquationis: dividatur data æquatio $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$ per hanc æquationem, & deprimetur data æquatio in duas alias, quarum una habet (m) dimensiones, altera vero $(n-m)$.

P R O B. XVII.

Datâ æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$, cujus quædam radices $(\alpha, \beta, \gamma, \delta, \epsilon, \&c.)$ datam habeant inter se relationem.

Invenire illas datæ æquationis radices $(\alpha, \beta, \gamma, \delta, \epsilon, \&c.)$, quæ datam inter se habeant relationem.

Scribantur in datâ æquatione pro radice (x) respectivæ radices $(\alpha, \beta, \gamma, \delta, \epsilon, \&c.)$ omnes inter se diversæ, & datam habentes relationem; & quantitatum resultantium communes divisores erunt radices quæsitæ.

Si numerus incognitarum quantitatum minor sit quam æquationum resultantium: methodus inveniendi communes divisores, ipsas quantitates deducit, quæ profecto non solummodo duarum quantitatum maximum communem divisorem invenit, sed quam plurimarum, si omnes divisores per minimum residuum continuo dividantur.

Ex. Sint radices datam inter se habentes relationem termini arithmeticæ seriei $a, a+b, a+2b, a+3b, \&c.$

Scribantur in datâ æquatione pro radice (x) termini prædicti $a, a+b, a+2b, a+3b, \&c.$ & resultant æquationes $a^n - pa^{n-1} + qa^{n-2} - ra^{n-3} + \&c. = 0$, $\overline{a+b}^n - p \times \overline{a+b}^{n-1} + q \times \overline{a+b}^{n-2} - r \times \overline{a+b}^{n-3} + \&c. = 0$, $\overline{a+2b}^n - p \times \overline{a+2b}^{n-1} + q \times \overline{a+2b}^{n-2} - r \times \overline{a+2b}^{n-3} + \&c. = 0$, &c. &c. &c.

& harum æquationum resultantium quicumque communis divisor radicem quæsitam præbebit.

Si data relatio inter (m) radices $(\alpha, \beta, \gamma, \delta, \epsilon, \&c.)$ datæ æquationis $(x^n -$

$(x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0)$ solummodo exprimi possit per æquationem hujusmodi $x^n - Px^{n-1} + Qx^{n-2} - Rx^{n-3} + Sx^{n-4} - \&c. = 0$. Assumatur data æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = w$ & $x^n - Px^{n-1} + Qx^{n-2} - Rx^{n-3} + Sx^{n-4} - \&c. = 0$, & problematis quinti ope inveniatur æquatio, cujus radix est (w), quæ sit $w^n - aw^{n-1} + bw^{n-2} - cw^{n-3} + dw^{n-4} - \&c. = 0$; fient coefficientes $a, b, c, d, e, \&c.$ nihilo respectivé æquales, & æquationum resultantium quicunque communis divisor radicem quæsitam præbet.

2. Est etiam alia methodus hoc problema resolvendi.

Sit data æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, & sint $\alpha, \beta, \gamma, \delta, \&c.$ ejus radices datam inter se habentes relationem.

Scribatur $x - \alpha = 0, x - \beta = 0, x - \gamma = 0, x - \delta = 0, \&c.$ & per æquationem ex horum factorum continuâ multiplicatione $\overline{x - \alpha} \times \overline{x - \beta} \times \overline{x - \gamma} \times \overline{x - \delta} \times \&c. = 0$ generatam dividatur data æquatio.

Singulus residui terminus nihilo æqualis fiat, & æquationum resultantium quicunque communis divisor erit radix quæsitæ.

Ex. 1. Sint α & β duæ datæ æquationis $x^n - px^{n-1} + \&c. = 0$ radices, quæ datam inter se habeant relationem, æquales, i. e. $\alpha = a$ & $\beta = a$: scribatur $x - a = 0$ & $x - a = 0$ & rectangulum $\overline{x - a} \times \overline{x - a} = x^2 - 2ax + a^2 = 0$; per hoc rectangulum data dividatur æquatio, & operatio est hujusmodi

$$\begin{array}{r} x^n - 2ax + a^2 \bigg) x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0 \left(x^{n-2} \right. \\ \underline{x^n - 2ax^{n-1} + a^2x^{n-2}} \\ 2a - px^{n-1} + q - a^2x^{n-2} \\ \underline{2a - px^{n-1} - 4a^2 - 2apx^{n-2} + 2a^3 - a^2px^{n-3}} \\ 3a^2 - 2ap + qx^{n-2} - 2a^3 - a^2p + rx^{n-3}, \&c. \end{array}$$

patet prioris residui termini coefficientem esse

$$\overline{n} a^{n-1} - \overline{n-1} p a^{n-2} + \overline{n-2} q a^{n-3} - \overline{n-3} r a^{n-4} + \&c.$$

posteriolem vero residui terminum esse

$$\overline{n-1} a^n - \overline{n-2} p a^{n-1} + \overline{n-3} q a^{n-2} - \overline{n-4} r a^{n-3} + \&c.$$

Fiant hæ duæ quantitates nihilo respectivé æquales, & duarum æqua-

æquationum resultantium quicunque communis divisor erit radix quæfita.

Ex. 3. Sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ termini arithmeticæ seriei $a, a+b, a+2b, \dots, a+m-1b$.

Scribantur pro summâ numerorum $(1, 2, 3, 4, \dots, m-1)$, pro summa rectangulorum sub singulis binis $(1 \times 2, 1 \times 3, 2 \times 3, 1 \times 4, 2 \times 4, 3 \times 4, 1 \times 5, 2 \times 5, \&c.)$ contentorum sub singulis ternis, quatuor, quinque, $\&c.$ $P, Q, R, S, T, \&c.$ respective; multiplicentur factores $x-a \times x-a-b \times x-a-2b \times x-a-3b \dots x-a-m-1b$ continuo in sese & contentum erit

$$\begin{aligned} x^m - m a x^{m-1} + m \times \frac{m-1}{2} a^2 x^{m-2} - m \times \frac{m-1}{2} \times \frac{m-2}{3} a^3 x^{m-3} + \&c. = 0. \\ -Pb \quad + \overline{m-1} \times Pba \quad - \overline{m-1} \times \frac{m-2}{2} Pba^2 \quad + \\ + Qb^2 \quad - \overline{m-2} Qb^2 a \quad + \\ - Rb^3 \quad + \end{aligned}$$

Per hoc contentum data dividatur æquatio, & residui terminorum quicunque communis divisor erit radix quæfita.

Ex. 4. Sint radices $\alpha, \beta, \gamma, \delta, \&c.$ geometricæ progressionis termini $(a, ra, r^2 a, r^3 a \dots r^{m-1} a.)$ Multiplicentur factores $x-a \times x-ra \times x-r^2 a \dots x-r^{m-1} a$ continuo in sese, & contentum erit

$$\begin{aligned} x^m - \frac{1-r^m}{1-r} \times a x^{m-1} + \frac{1-r^m \times 1-r^{m-1}}{1-r \times 1-r^2} \times r a^2 x^{m-2} \\ - \frac{1-r^m \times 1-r^{m-1} \times 1-r^{m-2}}{1-r \times 1-r^2 \times 1-r^3} \times r^3 \times a^3 x^{m-3} + \&c. = 0. \end{aligned}$$

Coefficiens termini (x^{m-s}) hujus æquationis erit æqualis fractioni, cujus numerator est

$$a^s \times r^{s \times \frac{s-1}{2}} \times \overline{1-r^{m-s+1}} \times \overline{1-r^{m-s+2}} \times \overline{1-r^{m-s+3}} \times \dots \times \overline{1-r^m}.$$

Denominator vero $\overline{1-r} \times \overline{1-r^2} \times \overline{1-r^3} \dots \overline{1-r^s}.$

Per hoc contentum data dividatur æquatio, & terminorum residui communis divisor erit radix quæfita.

Forſan haud multum abs re erit obſervare hinc oriri methodum inveniendi infinitam æquationem, quæ infinitas habet radices & poſſibiles & cognitæ.

Sit enim æquatio

$$1 - \frac{a}{1-r}x + \frac{ra^2}{1-r \times 1-r^2}x^2 - \frac{r^3a^3}{1-r \times 1-r^2 \times 1-r^3}x^3 + \frac{r^4a^4}{1-r \times 1-r^2 \times 1-r^3 \times 1-r^4}x^4 - \frac{r^{10}a^5}{1-r \times 1-r^2 \times 1-r^3 \times 1-r^4 \times 1-r^5}x^5 + \& \text{ in infinitum} = 0.$$

Et æquatio habet infinitas radices $\frac{1}{a}, \frac{1}{ra}, \frac{1}{r^2a}, \frac{1}{r^3a}, \frac{1}{r^4a}, \&c.$ in infinitum.

Et ſic aſſumi poſſint infinitæ conſimiles æquationes.

Hinc facile conſtabit, ut in plerisque caſibus hæc methodus difficiliorem præbet ſolutionem quam præcedens.

2. Si modo relatio inter (m) radices datæ æquationis exprimatur per æquationem $x^m - ax^{m-1} + bx^{m-2} - cx^{m-3} + \&c. = 0$. Methodus omnino eadem erit ac præcedens; i. e. dividatur data æquatio ($x^n - px^{n-1} + qx^{n-2} - \&c. = 0$) per æquationem $x^m - ax^{m-1} + bx^{m-2} - \&c. = 0$, quæ exprimit relationem inter (m) radices datæ æquationis, & terminorum reſidui quicunque communis diviſor (m) radices datæ æquationis quæſitas oſtendet.

Ex. 1. Sit data æquatio $x^4 - x^3 - 24x^2 + 4x + 80 = 0$, & æquatio exprimens relationem inter duas datæ æquationis radices $x^2 - 7x + 2 = 0$, dividatur prior æquatio per poſteriore, i. e.

$$x^4 - 7x^3 + 2(x^4 - x^3 - 24x^2 + 4x + 80) = x^2 + 6x + 18 - 2$$

& erit reſiduum $130 - 132x + 80 - 182 + 2^2$

Duorum reſultantium reſidui terminorum $130 - 132$ & $80 - 182 + 2^2$ inveniatur communis diviſor, & operatio erit $10 - 2(80 - 182 + 2^2) = 8 - 2$, & exinde $2 - 10 = 0$, & $2 = 10$; & conſequenter æquatio quæſita erit $x^2 - 7x + 10 = 0$, cujus radices datam habeant relationem, i. e. ſumma ejus radicum erit 7.

Paulo aliter reſolvi poteſt hoc exemplum. Pro (x) ſcribantur reſpective

spectivæ v & $7-v$, & resultant duæ æquationes $v^4 - v^3 - 24v^2 + 4v + 80 = 0$, & $v^4 - 27v^3 + 249v^2 - 893v + 990 = 0$. Harum duarum æquationum communis divisor erit radix quæsitæ.

Ex. 2. Sit data æquatio $x^6 + qx^4 - rx^3 + sx^2 - tx + v = 0$, sit æquatio exprimens relationem inter tres datæ æquationis radices $x^3 + ax - b = 0$: dividatur prior æquatio per posteriorem, i. e.

$x^3 + ax - b$) $x^6 + qx^4 - rx^3 + sx^2 - tx + v$) $x^3 + q - ax + b - r$
& residuum erit $s - qa + a^2x^2 + bq - t + ra - 2bax + b^2 - br + v$.

Horum trium residui terminorum $s - qa + a^2$, & $bq - t + ra - 2ba$ & $b^2 - br + v$, e methodo inveniendi communem divisorem sequitur nullum esse communem divisorem, & consequenter datam æquationem nullas habere dati generis radices, ni $q^2v - qrt + r^2s - 4sv + t^2 = 0$: si vero $q^2v - qrt + r^2s - 4sv + t^2 = 0$, tum æquationes quæsitæ erunt

$$x^3 + \sqrt{\frac{1}{4}q^2 - s + \frac{1}{2}q}x - \frac{1}{2}r + \sqrt{\frac{1}{4}r^2 - v} = 0.$$

Si vero $v = 0$, tum æquationes quæsitæ erunt

$$x^3 + \sqrt{\frac{1}{4}q^2 - s + \frac{1}{2}q}x - r = v, \text{ \& } x^3 + \sqrt{\frac{1}{4}q^2 - s + \frac{1}{2}q}x = 0.$$

Cor. 1. Sint plures æquationes, quam incognitæ quantitates, & inveniatur communis divisor unam incognitam quantitatem (x) solummodo involvens, & tot (n) possunt esse datæ formulæ æquationes, quot diversos valores habet incognita quantitas (x), quæ sit radix communis divisoris; ni duas, vel tres, vel plures radices æquales habet prædicta incognita quantitas, in quo casu forsan possunt esse $n+1$, $n+2$, vel plures datæ formulæ æquationes. Omnis valor (α) cujuscunque incognitæ quantitatis (x) habet unum & non plures valorem e singulis incognitis quantitibus sibi metipso (α) correspondentem; ni duo vel plures valores incognitæ quantitatis prædictæ (x) sint (α) inter se æquales; in quo casu tot possunt esse diversi valores e singulis incognitis quantitibus valori (α) radice (x) respondentes, quot valores incognitæ quantitatis (x) sint α .

Cor. 2. Sit numerus æquationum major quam numerus incognitarum quantitatum, & exterminentur omnes incognitæ quantitates; & si modo numerus æquationum major sit quam numerus incognitarum quantitatum per unitatem, resultat una solummodo æquatio cognitas quantitates involvens; si per duo, tres, &c. major sit; tum duæ, tres, &c. resultant æquationes cognitas quantitates solummodo involventes, quæ detegunt, quibus casibus possint esse æquationes hujusmodi.

E duabus, tribus, &c. æquationibus cognitas quantitates solummodo involventibus, facile formari possunt infinitæ aliæ.

Cor. 3. Datâ quâcunque quantitate e diversis irrationalibus quantitatibus ($A+B+C+D+E+\&c.$) conflata, invenire, utrum ullos admittat possibiles valores data quantitas, necne.

Affumantur pro radice irrationalis quantitatis A quantitas $a+p\sqrt{-1}$; pro radice quantitatis B quantitas $b+q\sqrt{-1}$; pro radicibus quantitatum irrationalium $C, D, E, \&c.$ respective $c+r\sqrt{-1}; d+s\sqrt{-1}; e+t\sqrt{-1}; \&c.$ & supponatur $p+q+r+s+t+\&c.=0$

Per problema inveniatur, utrum æquationes resultantes ex assumptis valoribus hanc habeant relationem ($p+q+r+s+t+\&c.=0$) inter se, & confit corollarium.

Hoc etiam investigari possit, supponendo datam quantitatem $=x$, & exterminando incognitas quantitates, & exinde inveniundo, utrum resultans æquatio possibiles admittat valores, necne.

Ex. 1. Sit quantitas data $\sqrt[2]{-3+4\sqrt{-1}} + \sqrt[3]{-16-16\sqrt{-1}}$: invenire utrum possibilem admittat valorem, necne.

Sint $\sqrt[2]{-3+4\sqrt{-1}}=a+p\sqrt{-1}$, & $\sqrt[3]{-16-16\sqrt{-1}}=b+q\sqrt{-1}$ & exinde resultant quatuor æquationes $a^2-p^2=-3$; $2ap=4$; $b^3-3bq^2=-16$, $3b^2q-q^3=-16$; supponatur etiam $p+q=0$, i. e. $q=-p$; & per methodum prædictam inveniantur $a=1, p=2, b=2, q=-2$, ergo admittet possibilem valorem.

Si

nam reductis, facile constat (n) esse diversas radices. Æquatio, cujus radix sit α , erit

$$\left. \begin{aligned} \alpha^n - a \times \alpha^{n-1} + b \\ - n \end{aligned} \right\} \left. \begin{aligned} \alpha^{n-2} - c \\ + n-1 a \end{aligned} \right\} \left. \begin{aligned} \alpha^{n-3} + \frac{d}{n-2} \\ + n \times \frac{n-3}{2} \end{aligned} \right\} \left. \begin{aligned} \alpha^{n-4} - e \\ + n-3 \times c \\ - n-1 \times \frac{n-4}{2} a \end{aligned} \right\} \alpha^{n-5} \\ \left. \begin{aligned} + f \\ - n-4 d \\ + n-2 \times \frac{n-5}{2} b \\ - n \times \frac{n-4}{2} \times \frac{n-5}{3} \end{aligned} \right\} \left. \begin{aligned} \alpha^{n-6} - g \\ + n-5 e \\ - n-3 \times \frac{n-6}{2} c \\ + n-1 \times \frac{n-5}{2} \times \frac{n-6}{3} a \end{aligned} \right\} \left. \begin{aligned} \alpha^{n-7} + b \\ - n-6 f \\ + n-4 \times \frac{n-7}{2} d \\ - n-2 \times \frac{n-6}{2} \times \frac{n-7}{3} \\ + n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} \end{aligned} \right\} \alpha^{n-8}$$

Lex, quam observat series, quæ exprimat coefficientem termini (α^{n-m}) hujus seriei, vel literas, vel earum coefficientes respicit.

Literæ erunt coefficientes terminorum x^{2n-m} , x^{2n-m+2} , x^{2n-m+4} , x^{2n-m+6} , x^{2n-m+8} , &c. datæ æquationis.

Coefficiens cujuscunque literæ, quæ fit coefficiens termini $x^{2n-m+2r}$ datæ æquationis, erit $\frac{n-m+1}{n-m+2} \times \frac{n-m+2}{2} \times \frac{n-m+3}{3} \times \frac{n-m+4}{4} \times \frac{n-m+5}{5} \times \dots \times \frac{n-m+r-1}{r}$: Si vero r fit 0, tum coefficiens erit 1; si r fit 1, tum coefficiens erit $n-m+2$.

Signum coefficienti literæ (quæ fit coefficiens termini $x^{2n-m+2r}$) affixum erit + vel -, prout $m+r$ fit par vel impar numerus.

Cor. Sit æquatio prædicta $x^{2n} \pm x^{2n-1} + x^{2n-2} \pm x^{2n-3} + x^{2n-4} \mp \&c. = 0$: & in quadraticam $x^2 + vx + 1 = 0$ reduci potest æquationis ope v^n

$$\begin{aligned} &= v^{n-1} - \frac{n-1}{n-2} v^{n-2} + \frac{n-2}{n-3} v^{n-3} - \frac{n-3}{n-4} v^{n-4} + \frac{n-4}{n-5} v^{n-5} - \frac{n-5}{n-6} v^{n-6} + \frac{n-6}{n-7} v^{n-7} - \frac{n-7}{n-8} v^{n-8} + \&c. = 0. \end{aligned}$$

Cor.

$$\begin{aligned}
 & \left. \begin{aligned}
 & \frac{n-7}{3} \times \frac{n-7}{4} \\
 & \frac{n-7}{b}
 \end{aligned} \right\} a^{n-3} \\
 & \begin{aligned}
 & + \frac{n-7}{5} \times \frac{n-8}{2} \\
 & + \frac{n-3}{2} \times \frac{n-7}{2} \times \frac{n-8}{3} \\
 & - \frac{n-1}{2} \times \frac{n-6}{2} \times \frac{n-7}{3} \times \frac{n-8}{4}
 \end{aligned} \\
 & \left. \begin{aligned}
 & \frac{n-8}{c} \\
 & \frac{n-8}{a}
 \end{aligned} \right\} a^{n-2} + \&c. = 0.
 \end{aligned}$$

Si quantitas irrationalis fit $\sqrt[n]{a + \sqrt{-b^2}} + \sqrt[n]{a - \sqrt{-b^2}}$; tum facile constat, si modo $a + p\sqrt{-1}$ fit radix quantitatis $\sqrt[n]{a + \sqrt{-b^2}}$, $a - p\sqrt{-1}$ etiam esse radicem quantitatis $\sqrt[n]{a - \sqrt{-b^2}}$, & possibiles esse valores.

Cor. 1. In plerisque hujusmodi casibus e data relatione inter radices deduci potest maximus radicum numerus, quas habere potest quæcumque (n) dimensionum æquatio: sed minime generaliter dici potest, ni quis prædictâ utatur methodo communes divisiore inveniendi, utrum ullas radices dati generis habeat data æquatio, necne.

Cor. 2. Quædam æquationes datæ formulæ necessario habeant radices, quæ assignabilem inter se habeant relationem: In his autem casibus haud plures resultant diversæ æquationes, quam incognitæ quantitates: ergo ex æquationibus resultantibus dici potest, quot radices, quæ assignabilem inter se habeant relationem, data recipiat æquatio.

Ex. 1. Sit æquatio

$x^m + Px^{n-1}x^m + Qx^{n-2}x^m + Rx^{n-3}x^m + Sx^{n-4}x^m + \&c. = 0$: Hæc æquatio necessario habet radices, quæ eandem relationem inter se habeant, quam habent radices æquationis $x^m + p = 0$, $x^m + q = 0$, $x^m + r = 0$, $x^m + s = 0$, &c. quarum numerus sit (n); ducantur enim hæ (n) æquationes in sese & resultat æquatio ($n \times m$) dimensionum: fiant termini æquationis resultantis & datæ respective inter se æquales, & facile constat (n) esse æquationes, quarum radices assignabilem habent inter se relationem.

Ex. 2. Sit æquatio $x^{2n} + ax^{2n-1} + bx^{2n-2} + cx^{2n-3} + dx^{2n-4} + ex^{2n-5} + fx^{2n-6} + gx^{2n-7} + hx^{2n-8} + kx^{2n-9} \dots px^n \dots kx^9 + bx^8 + gx^7 + fx^6 + ex^5 + dx^4 + cx^3 + bx^2 + ax + 1 = 0$. & æquatio $x^2 + ax + 1 = 0$ necessario exprimet relationem inter duas datæ æquationis radices: i. e. duæ radices in sese multiplicatæ, faciunt rectangulum $= 1$; si enim (n) diversæ hujusce generis æquationes in sese ducantur, & termini resultantis & datæ æquationis respective æquales inter se supponantur, resultabunt (n) æquationes tot (n) incognitas quantitates habentes, quibus in u-

nam

Cor. Tres radices æquationis $x^3 + 1 = 0$ erunt respectivo -1 , $\frac{-1 \pm \sqrt{-3}}{2}$; quatuor radices æquationis $x^4 + 1 = 0$ erunt $\pm \sqrt{\frac{-1}{2}}$, $\pm \sqrt{\frac{-1}{2}}$; quinque radices æquationis $x^5 + 1 = 0$ erunt -1 , $\frac{-1 \pm \sqrt{1 \pm \frac{1}{2}}}{2}$, $\frac{-1 \pm \sqrt{-2 \pm \sqrt{1 \pm \frac{1}{2}}}}{2}$; sex vero radices æquationis ($x^6 + 1 = 0$) erunt $\pm \sqrt{-1}$, $\frac{\pm \sqrt{3} \pm \sqrt{-1}}{2}$; septem vero radices æquationis $x^7 + 1 = 0$ haud exprimi possunt per quadraticas radices.

Ex. 3. Sit æquatio $x^{2n+1} + ax^{2n} + bx^{2n-1} + cx^{2n-2} + dx^{2n-3} + ex^{2n-4} + fx^{2n-5} + gx^{2n-6} + bx^{2n-7} + kx^{2n-8} \dots + px^{n+1} + px^n \dots kx^9 + bx^8 + gx^7 + fx^6 + ex^5 + dx^4 + cx^3 + bx^2 + ax + 1 = 0$. Hujus æquationis una radix erit -1 , & (n) etiam erunt quadraticæ æquationes hujusce formulæ $x^2 + \alpha x + 1 = 0$, $x^2 + \beta x + 1 = 0$, $x^2 + \gamma x + 1 = 0$; &c. æquatio, cujus radix est α , erit

$$\left. \begin{array}{l} \alpha^n - \alpha - 1 \times \alpha^{n-1} + b \\ -a \\ -n-1 \end{array} \right\} \left. \begin{array}{l} \alpha^{n-2} - c \\ +b \\ +n-2 \end{array} \right\} \left. \begin{array}{l} \alpha^{n-3} + d \\ -c \\ +n-3 \\ +n-2 \times \frac{n-3}{2} \end{array} \right\} \left. \begin{array}{l} \alpha^{n-4} - e \\ +d \\ +n-4 \\ -n-4 \times \frac{n-4}{2} \\ +n-3 \times \frac{n-4}{2} \end{array} \right\} \left. \begin{array}{l} \alpha^{n-5} + f \\ -e \\ +n-5 \\ +n-4 \times \frac{n-5}{2} \\ -n-4 \times \frac{n-5}{2} \\ -n-3 \times \frac{n-4}{2} \times \frac{n-5}{3} \end{array} \right\} \alpha^{n-6} - \&c. = 0.$$

Lex, quam observat series, quæ exprimat coefficientem termini (α^{n-m}) hujus seriei, vel literas, vel earum coefficientes, respicit.

Literæ erunt coefficientes terminorum x^{2n-m+1} , x^{2n-m+2} , x^{2n-m+3} , &c. datæ æquationis.

Coefficiens cujuscunque literæ, quæ sit coefficiens termini $x^{2n-m+2r+1}$ datæ æquationis, erit $\frac{n-m+1}{n-m+1} \times \frac{n-m+2}{2} \times n-m$

$\times \frac{n-m+3}{3} \dots \frac{n-m+r}{r}$. si r sit 0, coefficientis erit 1.

Signum coefficienti literæ (quæ vel sit coefficientis termini $x^{2n-m+2r+1}$, vel termini $x^{2n-m+2r}$) affixum, erit + vel —, prout $m+r$ sit par vel impar numerus.

PROB.

Invenire æquationis dimensiones, cujus ope altera ($x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$) n dimensionum in alteram ($x^m - ax^{m-1} + bx^{m-2} - cx^{m-3} + \&c. = 0$) m dimensionum reduci potest, posito (n) majore quam m, & radicibus reductæ æquationis $x^m - ax^{m-1} + bx^{m-2} - cx^{m-3} + \&c. = 0$ etiam esse radices æquationis $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, ex quâ reducta fuit.

Dimensiones requisitæ erunt æquales fractioni

$$n \times \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \dots \frac{n-m+1}{m}.$$

DEMONSTRATIO.

Æquatio $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$ habet (n) radices, & ex quavis combinatione (m) radicum hujus æquationis formari potest æquatio (m) dimensionum, quæ respondebit problemati; ergo quot sunt combinationes (m) radicum in majore multitudine (n) radicum, tot erunt problematis solutiones, & consequenter tot erunt radices æquationis reducentis.

Numerus autem combinationum, quibus quantitatuum vel radicum multitudo (m) in majore multitudine (n) esse possit; æqualis erit fractioni $n \times \frac{n-1}{2} \cdot \frac{n-2}{3} \dots \frac{n-m+1}{m}$. ergo numerus radicum, & consequenter dimensiones æquationis reducentis æqualis erit fractioni

$$n \times \frac{n-1}{2} \cdot \frac{n-2}{3} \dots \frac{n-m+1}{m}. \text{ Q. E. D.}$$

Haud

Haud refert, an requiretur investigatio coefficientis a , vel b , vel c , &c. tot enim diversos valores habent coefficientes vel a , vel b , vel c , &c. & consequenter eodem erunt æquationum dimensiones, quarum radices sunt vel a , vel b , vel c , &c. respective.

Datâ autem unâ coefficiente, quâ substitutâ pro suo valore, e præcedenti inveniendi methodo, utrum datæ æquationis radices datam inter se habeant relationem, *i. e.* e methodo communes divisores inveniendi, erui possunt reliquæ incognitæ quantitates. Si vero duo valores datæ coefficientis sint æquales, tum reliquæ incognitæ quantitates prædictâ methodo & quadraticarum æquationum resolutionum ope invenientur; si tres valores sint æquales, tum reliquæ incognitæ quantitates prædictâ methodo & cubicarum æquationum resolutionum ope investigari possint, & sic deinceps.

Hoc in loco observetur, quod postea generaliter docebitur, omnes incognitas quantitates a , b , c , &c. eandem habere irrationalitatem; ni duo vel plures valores unius incognitæ quantitatæ sint inter se æquales; si duo uniuscujusque quantitatæ valores (α) sint inter se æquales, tum resultabunt quadraticæ æquationes, quarum radices sint reliquæ incognitæ quantitates, & quæ solummodo habebunt irrationalitatem, quam habet valor (α) prædictæ incognitæ quantitatæ; si tres valores (α) sint inter se æquales, tum resultant confimiles cubicæ æquationes; & sic deinceps.

Ex. Biquadratica $x^4 - px^3 + qx^2 - rx + s = 0$ reduci potest in quadraticam $x^2 - Px + Q = 0$, æquationis ($4 \times \frac{4-1}{2} = 6$) dimensionum ope.

$$\text{Æquatio, cujus radix est } P, \text{ erit } P + \frac{p^6}{4} + 2q - \frac{3}{4}p^2 \times P + \frac{p^4}{4} + q^2 - 4s - qp^2 + rp + \frac{3}{16}p^4 \times P + \frac{p^2}{4} - \frac{p^3}{8} - \frac{qp}{2} + r = 0.$$

Sit $p = 0$, *i. e.* summa quatuor radicum ($\alpha, \beta, \gamma, \delta$) datæ æquationis fit $= 0$; summa autem duarum ($\alpha + \beta$) est P , ergo summa duarum

rum reliquarum ($\gamma + \delta$) erit $-P = -\alpha - \beta$; & tres radices æquationis, cujus radix est P , erunt affirmativæ; tres vero negativæ & affirmativis æquales; & consequenter ejus formula quoad resolutionem erit cubica.

Æquatio, cujus radix est P , erit

$$P^6 + 2qP^4 + q^2 - 4sP^2 - r^2 = 0.$$

Hujusce æquationis radices ($\alpha + \beta$, $\alpha + \gamma$, $\alpha + \delta$, $\beta + \gamma$, $\beta + \delta$, $\gamma + \delta$) sint π , ρ , σ , τ , υ , ϕ respective; & radices datæ æquationis erunt $\frac{\pi + \rho - \tau}{2}$, $\frac{\pi + \tau - \rho}{2}$, $\frac{\tau + \rho - \pi}{2}$, $\frac{\sigma + \upsilon - \pi}{2}$ respective.

Data unâ radice (π) quantitatis P , tum quatuor datæ æquationis radices erunt $\pm \sqrt{\frac{-\pi^3 - 2q\pi - 2r}{2\pi}} + \frac{1}{2}\pi$, $\pm \sqrt{\frac{-\pi^3 - 2q\pi + 2r}{2\pi}} + \frac{1}{2}\pi$.

Cor. 1. Hæc resolutio solummodo generalis erit, cum duæ radices datæ æquationis sint possibiles, duæ vero impossibiles: Hoc enim in casu cubicæ æquationis $P^3 + 2qP^2 + q^2 - 4sP - r^2 = 0$ una radix erit possibilis, duæ vero impossibiles; in quo casu solummodo inveniri possint per generalem methodum cubicæ æquationis radices.

Æquatio, cujus radix est $\mathcal{Q}$ rectangulum sub quibusque duabus biquadraticæ æquationis $x^4 + qx^2 - rx + s = 0$ radicibus, erit $\mathcal{Q}^6 - q\mathcal{Q}^5 - s\mathcal{Q}^4 + 2sq - r^2\mathcal{Q}^3 - s^2\mathcal{Q}^2 - qs^2\mathcal{Q} + s^3 = 0$.

Cor. 2. Hæc vero methodo æquatio reducens plures habet dimensiones quam data æquatio, & haud generaliter admittat divisores, nec ejus formula ea est, cujus resolutio cognoscitur: frustra esset igitur per talem reductionem æquationum resolutionem generalem quærere.

L E M M A.

Omnis biquadratica æquatio hujusce formulæ $x^2 - 2 \times a + b \sqrt{-1}$ $x - c - d \sqrt{-1} = 0$ semper distingui potest in duas rationales quadraticas æquationes.

M

Sint

$$\text{Sint enim } \alpha^2 = \frac{a^2 - b^2 + c + \sqrt{a^2 - b^2 + c^2 + 2ab + d^2}}{2}$$

$$\beta^2 = \frac{+ \sqrt{a^2 - b^2 + c^2 + 2ab + d^2} - a^2 + b^2 - c}{2}, \text{ ubi } \alpha$$

& β possibiles quantitates semper denotent.

Rationales quadraticæ æquationes erunt respectivæ

$$x^2 - 2a + \alpha x + a + \alpha^2 + \beta + b^2 = 0, \text{ \& } x^2 - 2a - \alpha x + a - \alpha^2 + \beta - b^2 = 0.$$

THEOR. VII.

Omnis æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, vel componitur e simplicibus divisoribus in sese ductis ($x - \alpha \times x - \beta \times x - \gamma \times \&c.$), quæ solummodo possibiles quantitates ($\alpha, \beta, \gamma, \delta, \&c.$) continent, vel e simplicibus & quadraticis divisoribus $x - \alpha, x - \beta, x - \gamma, \&c. x^2 - \pi x + P, x^2 - \rho x + R, x^2 - \sigma x + S, \&c.$ in sese ductis, qui solummodo possibiles quantitates continent.

DEMONSTRATIO.

Omnis æquatio $x^{2n+1} - px^{2n} + qx^{2n-1} - \&c. = 0$, cujus dimensiones ($2n+1$) sunt impares, unam saltem habet possibilem divisorem ($x - \alpha$), cujus α sit possibilis quantitas, & hujus ope deprimi potest data æquatio in alteram parium ($2n$) dimensionum.

Omnis æquatio $x^{4n+2} - px^{4n+1} + \&c. = 0$ dimensionum ($4n+2$) deprimi potest in quadraticam æquationis $\frac{4n+2}{2} \times \frac{4n+1}{2} = 2n+1 \times \frac{4n+1}{2}$ dimensionum ope: una autem radix æquationis reducentis, cujus dimensiones sint $2n+1 \times 4n+1$, erit possibilis quantitas; ergo una quadratica æquatio, quæ solummodo possibiles habeat quantitates, erit divisor datæ æquationis.

Omnis

Omnia æquatio $x^{8n+4} - px^{8n+3} + qx^{8n+2} - \&c. = 0$ dimensionum $(8n+4)$ deprimi potest in quadraticam æquationis $\overline{8n+4} \times \frac{8n+3}{2} = \overline{4n+2} \times \overline{8n+3}$ dimensionum ope: Hæc autem æquatio unam habet quadraticam æquationem, cujus duæ coefficientes sint possibiles quantitates, quæ sit $x^2 - ax + b = 0$; cujus radices sunt $\frac{a \pm \sqrt{a^2 - 4b}}{2}$.: fed æquatio $(8n+4)$ dimensionum deprimi potest in quadraticam utrarum harum radicum ope, quibus substitutis, constat æquationem saltem habere biquadraticam æquationem hujusce formulæ $x^2 + A + c\sqrt{a^2 - 4b}x + B + d\sqrt{a^2 - 4b} = 0$, ubi A, B, c, d possibiles denotent quantitates: fed per lemma præcedens hæc biquadratica conficitur e duabus rationalibus quadraticis; ergo æquatio $(8n+4)$ dimensionum saltem habet duas rationales quadraticas.

Eodem prorsus modo probari potest æquationem $x^{16n+8} - px^{16n+7} + qx^{16n+6} - \&c. = 0$ saltem habere quatuor possibiles quadraticas æquationes: & eodem modo constat æquationem $x^{32n+16} - px^{32n+15} + \&c. = 0$ saltem habere octo possibiles quadraticas æquationes, & sic deinceps: ergo constat omnes æquationes habere vel simplicem, vel quadraticas possibiles æquationes: dividatur data æquatio per simplicem vel quadraticas possibiles æquationes, & deprimatur in alteram pauciores dimensiones habentem: æquationis autem resultantis possibiles erunt simplices vel quadraticæ æquationes, ergo adhuc deprimi possit resultans æquatio; & sic deinceps; unde constat theorema.

Æquationes reduci possint, si ita in duas æquales partes dividantur, ut ex utrâque parte eadem radix extrahatur.

P R O B. XIX.

Biquadraticam $x^4 + 2px^3 = qx^2 + rx + s$ in quadraticam ope cubicæ reducere.

M 2

Addatur

Addatur ad utramque æquationis partem $\overline{p^2 + 2nx^2 + 2pnx + n^2}$, & prodit

$$x^4 + 2px^3 + \overline{p^2 + 2nx^2 + 2pnx + n^2} = \overline{p^2 + 2n + qx^2 + 2pn + rx + s + n^2}.$$

Supponatur $4 \times s + n^2 \times \overline{p^2 + 2n + q} = \overline{2pn + r^2}$: Reducatur hæc æquatio, & transponantur termini, & fit cubica

$$8n^3 + 4qn^2 + 8s - 4rp \times n + 4qs + 4p^2s - r^2 = 0.$$

Hinc inveniatur incognita quantitas (n), quâ pro suo valore (n) in æquatione $x^4 + 2px^3 + \overline{p^2 + 2nx^2 + 2pnx + n^2} = \overline{p^2 + 2n + qx^2 + 2pn + rx + s + n^2}$ substitutâ, & extractâ utrobique radice quadraticâ, resultat æquatio quadratica $x^2 + px + n = \sqrt{p^2 + 2n + q}x + \sqrt{s + n^2}$.

Cor. 1. Biquadraticæ æquationis quatuor radices (si modo π sit radix cubicæ æquationis $8n^3 + 4qn^2 + 8s - 4rp \times n + 4qs + 4p^2s - r^2 = 0$) erunt respective radices duarum quadraticarum $x^2 + p \left(\frac{\pi + \sqrt{p^2 + 2\pi + q}}{\pi - \sqrt{s + \pi^2}} \right) x + \pi - \sqrt{s + \pi^2} = 0$, & $x^2 + p \left(\frac{\pi - \sqrt{p^2 + 2\pi + q}}{\pi + \sqrt{s + \pi^2}} \right) x + \pi + \sqrt{s + \pi^2} = 0$,

duæ etiam aliæ inveniuntur quadraticæ æquationes ex hac resolutione $x^2 + p \left(\frac{\pi + \sqrt{p^2 + 2\pi + q}}{\pi - \sqrt{s + \pi^2}} \right) x + \pi + \sqrt{s + \pi^2} = 0$, & $x^2 + p \left(\frac{\pi - \sqrt{p^2 + 2\pi + q}}{\pi + \sqrt{s + \pi^2}} \right) x + \pi - \sqrt{s + \pi^2} = 0$.

i. e. si modo quatuor radices datæ æquationis sint $\alpha, \beta, \gamma, \delta$: & duæ priores quadraticæ æquationes sint $x^2 - \overline{\alpha + \beta}x + \alpha\beta = 0$, & $x^2 - \overline{\gamma + \delta}x + \gamma\delta = 0$; duæ posteriores erunt $x^2 - \overline{\alpha + \beta}x + \gamma\delta = 0$, & $x^2 - \overline{\gamma + \delta}x + \alpha\beta = 0$.

Si vero $p = 0$, tum hæc quatuor quadraticæ æquationes erunt divisores duarum biquadraticarum æquationum $x^4 - qx^2 - rx - s = 0$, & $x^4 - qx^2 + rx - s = 0$: i. e. resolutio ex hac methodo deducta præbet omnes datæ æquationis radices, & omnes earum negativas, E. G. fit biquadratica æquatio $x^4 - 25x^2 + 60x - 36 = 0$; pro p, q, r, s scribantur

scribantur 0, 25, -60, 36 respective in reducenti æquatione $8n^3 + 4qn^2 + 8s - 4rp \times n + 4qs + 4p^2s - r^2 = 0$, & fit $8n^3 + 100n^2 + 288n = 0$; cujus radices sunt respective 0, -8, $-\frac{9}{2}$; quibus pro suo valore (n) in æquatione $x^2 + 2px + n = \sqrt{q + p^2 + 2n}x + \sqrt{s + n^2}$ substitutis, resultant tres æquationes $x^2 = \sqrt{25}x + \sqrt{36}$, & $x^2 - 8 = \sqrt{9}x + \sqrt{100}$, & $x^2 - \frac{9}{2} = \sqrt{16}x + \sqrt{\frac{225}{2}}$. Ex singulis his tribus æquationibus radices vel sunt 1, 2, 3, -6; qui numeri sunt radices datæ æquationis $x^4 - 25x^2 + 60x - 36 = 0$, vel earum negativi -1, -2, -3, +6.

Cor. 2. Sint $\alpha, \beta, \gamma, \delta$ respectivæ datæ æquationis $x^4 + 2px^3 - qx^2 - rx - s = 0$ radices, & tres radices æquationis reducentis $8n^3 + 4qn^2 + 8s - 4rp \times n + 4qs + 4p^2s - r^2 = 0$ erunt respective $\frac{\alpha\beta + \gamma\delta}{2}$, $\frac{\alpha\gamma + \beta\delta}{2}$, $\frac{\alpha\delta + \beta\gamma}{2}$; & quantitatis $\sqrt{q + p^2 + 2n}$ sex valores erunt respective $\frac{\alpha + \beta - \gamma - \delta}{2}$, $\frac{\gamma + \delta - \alpha - \beta}{2}$, $\frac{\alpha + \gamma - \beta - \delta}{2}$, $\frac{\beta + \delta - \alpha - \gamma}{2}$, $\frac{\alpha + \delta - \beta - \gamma}{2}$, $\frac{\beta + \gamma - \alpha - \delta}{2}$; sex vero valores quantitatis $\sqrt{s + n^2}$ erunt respective $\frac{\alpha\beta - \gamma\delta}{2}$, $\frac{\gamma\delta - \alpha\beta}{2}$, $\frac{\alpha\gamma - \beta\delta}{2}$, $\frac{\beta\delta - \alpha\gamma}{2}$, $\frac{\alpha\delta - \beta\gamma}{2}$, $\frac{\beta\gamma - \alpha\delta}{2}$.

Cor. 3. Cum vero inventi fuerint omnes diversi valores quantitatis $\sqrt{p^2 + 2n + q}$, sine ulteriori radicum extractione investigari possint datæ æquationis radices: sint enim $a = \frac{\alpha + \beta - \gamma - \delta}{2}$, $b = \frac{\alpha + \gamma - \beta - \delta}{2}$, $c = \frac{\alpha + \delta - \beta - \gamma}{2}$, & erit $\alpha = \frac{+a + b + c - p}{2}$, $\beta = \frac{+a - b - c - p}{2}$, $\gamma = \frac{+b - a - c - p}{2}$, $\delta = \frac{+c - a - b - p}{2}$: & sic de cæteris quantitativis.

Cor.

Cor. 4. Ex hac regulâ haud inveniri possint radices datæ biquadraticæ, ni duæ radices sint possibiles, duæ vero impossibiles: si enim nullæ vel quatuor radices datæ æquationis sint impossibiles, tum reducens æquatio, quæ sit cubica, tres possibiles habet radices; & exinde constat, ut nulla existat regula, e qua generaliter inveniri possint ejus radices.

Cor. 5. Cum supponatur in resolutione problematis $2 \times \sqrt{p^2 + 2n + q} \times \sqrt{s + n^2} = 2pn + r$, & exinde recte concluditur $4 \times \sqrt{p^2 + 2n + q} \times \sqrt{s + n^2} = 2pn + r^2$: supposito autem $4 \times \sqrt{p^2 + 2n + q} \times \sqrt{s + n^2} = 2pn + r^2$, minime sequitur omnem valorem rectanguli $2 \sqrt{p^2 + 2n + q} \times \sqrt{s + n^2}$ æqualem esse $2pn + r$: quot enim valores rectanguli prædicti sint $2pn + r$, tot erunt $2pn - r$; & consequenter exinde in resolutionem irrepsertunt e singulo valore quantitatis (n) duo valores præter quæsitos in quantitatem $\sqrt{p^2 + 2n + q} \times \sqrt{s + n^2}$; & exinde quatuor radices in resolutionem: unde tribus valoribus quantitatis (n) substitutis in datam resolutionem irrepsertunt duodecim radices, & resolutio erit resolutio æquationis (16) dimensionum, cujus quatuor radices erunt quatuor datæ æquationis radices.

Ex. 2. Æquationem quadrato-cubicam ($x^5 + qx^3 + rx^2 + sx + t = 0$) in cubicam reducere.

Multiplicetur data æquatio per (x), & resultat $x^6 + qx^4 + rx^3 + sx^2 + tx = 0$; addatur ad utramque æquationis partem $kx^3 + dx + b^2$, & prodit $kx^3 + dx + b^2 = k^2 + 1 x^6 + 2kd + q x^4 + 2kb + r x^3 + d^2 + s x^2 + 2db + t x + b^2$. Supponantur $2 \sqrt{k^2 + 1} \times b = 2kb + r$; $2 \sqrt{k^2 + 1} \times \sqrt{d^2 + s} = 2kd + q$; $2 \sqrt{d^2 + s} \times b = 2db + t$; quibus reductis, resultant tres æquationes $4b^2 = 4rkb + r^2$; $4sk^2 + 4d^2 + 4s = 4qkd + q^2$; $4sb^2 = 4tdb + t^2$; e primâ æquatione sequitur $k = \frac{4b^2 - r^2}{4rb}$, e tertiâ vero $d = \frac{4sb^2 - t^2}{4tb}$; quibus quantitibus pro suis valoribus in secundâ æquatione substitutis, resultat æquatio biquadratica formulæ vero quadraticæ,

$$\text{raticæ } 4 \times \frac{4sb^2 - t^2}{16t^2b^2} + 4s \times \frac{4b^2 - r^2}{16r^2b^2} + 4s = 4q \times \frac{4b^2 - r^2}{4rb} \times \frac{4sb^2 - t^2}{4tb} + q^2:$$

reducatur hæc æquatio, & fit $sr^2 + t^2 - qtr \times 16sb^4 + 4grtb^2 + r^2t^2 = 0$.

Cor. 1. Hæc resolutio solummodo inveniet radices datæ æquationis, cum $sr^2 + t^2 - qtr = 0$: si vero $sr^2 + t^2 - qtr = 0$, assumatur pro literâ (b) quæcunque quantitas, & capiantur $k = \frac{4b^2 - r^2}{4rb}$; & $d = \frac{4sb^2 - t^2}{4tb}$; tum data æquatio erit (differentia inter duo quadrata)

$$\left[\frac{4b^2 + r^2}{4rb} x^3 + \frac{4sb^2 + t^2}{4tb} x + b \right]^2 - \left[\frac{4b^2 - r^2}{4rb} x^3 + \frac{4sb^2 - t^2}{4tb} x + b \right]^2 = x^6$$

+ $qx^4 + rx^3 + sx^2 + tx = 0$; quæ quidem quadrata infinitis modis variari possunt, quoniam pro litera (b) scribatur quæcunque quanti-

tas: e.g. pro $2b$ scribatur r ; & evanescet terminus $\frac{4b^2 - r^2}{4rb} x^3$, & $\frac{4b^2 + r^2}{4rb}$

fit 1; æquationes autem depressæ erunt $\frac{4b^2 + r^2}{4rb} x^3 + \frac{4sb^2 + t^2}{4tb} x + b$

$$= \pm \frac{4b^2 - r^2}{4rb} x^3 + \frac{4sb^2 - t^2}{4tb} x + b.$$

Cor. 2. Æquatio autem $16sb^4 + 4grtb^2 + r^2t^2 = 0$ hinc oritur: supposito $4k^2 + 1 \times b^2 = 2kb + r^2$, concluditur $2b\sqrt{k^2 + 1} = 2kb + r$; duo enim sunt diversi valores quantitatis $2b\sqrt{k^2 + 1}$; alter, qui problema resolvit; alter vero minime; e posterioris generis radicibus resultat prædicta æquatio.

Ex. 3. Æquationem cubo-cubicam ($x^6 + qx^4 + rx^3 + sx^2 + tx + v = 0$) in cubicam reducere.

Addatur ad utramque æquationis partem $\overline{kx^3 + dx + b^2}$, & prodit $\overline{kx^3 + dx + b^2} = \overline{k^2 + 1} x^6 + \overline{2kd + q} x^4 + \overline{2kb + r} x^3 + \overline{d^2 + s} x^2 + \overline{2db + t} x + \overline{b^2 + v}$: Supponantur $2\sqrt{k^2 + 1} \times \sqrt{s + d^2} = q + 2kd$; $2\sqrt{k^2 + 1} \times \sqrt{v + b^2} = r + 2kb$; & $2\sqrt{s + d^2} \times \sqrt{v + b^2} = t + 2db$; quibus reductis resultant tres æquationes $4vk^2 + 4b^2 + 4v = 4rkb + r^2$;

$4sk$

$$4sk^2 + 4d^2 + 4s = 4qkd + q^2; \text{ \& } 4vs + 4vd^2 + 4sb^2 = 4tdb + t^2;$$

e primâ æquatione sequitur $k = \frac{\sqrt{r^2 - 4v} \times \sqrt{v + b^2} + rb}{2v}$, e tertiâ

$$d = \frac{\sqrt{r^2 - 4vs} \times \sqrt{v + b^2} + tb}{2v};$$

quibus in secundâ æquatione pro suis valoribus substitutis, resultat æquatio, quâ reductâ, ita ut exterminetur irrationalis quantitas $\sqrt{v + b^2}$, prodit biquadratica formulæ vero quadraticæ: si vero exterminentur omnes irrationales quantitates, prodit æquatio octo dimensionum, formulæ vero biquadraticæ. Hæ autem æquationes resultantes multiplicabuntur in $q^2v - qrt + r^2s - 4sv + t^2$.

Cor. Hæc resolutio solummodo inveniet radices datæ æquationis, cum quantitas $q^2v - qrt + r^2s - 4sv + t^2 = 0$. Hoc enim in casu trium radicum resultantis cubicæ æquationis summa nihilo erit æqualis, quod in primâ substitutione revera hujus & præcedentis exempli pro concessio assumitur: pro litera b assumi potest quæcunque quantitas; & exinde consequuntur valores quantitatum k & d correspondentes: resultantis autem æquationis radices, quæ sint b , ex eodem fonte ac in præcedenti exemplo suum originem duxere.

Ex. 4. Sit æquatio

$$x^{2n} + qx^{2n-1} - rx^{2n-3} + sx^{2n-4} - tx^{2n-5} + vx^{2n-6} \text{ \&c. } = 0,$$

$$\text{assumatur } x^n + ax^{n-2} + bx^{n-3} + cx^{n-4} + \text{ \&c. } = ax^{n-1} + \beta x^{n-2} + \gamma x^{n-3} + \text{ \&c. }$$

$$\text{\& exinde } x^{2n} + 2a - \alpha^2 x^{2n-2} + 2b - 2\alpha\beta x^{2n-3} + 2c + a^2 - 2\alpha\gamma - \beta^2 x^{2n-4} + \text{ \&c. } = 0.$$

Fiant correspondentes hujus & datæ æquationis coefficientes inter se æquales, i. e. $2a - \alpha^2 = q$, $2b - 2\alpha\beta = -r$, $2c + a^2 - 2\alpha\gamma - \beta^2 = s$, &c. & reducantur æquationes, ita ut exterminentur omnes præter unam incognitam quantitatem, & resultat æquatio reducens; sed hæc æquatio reducens plures habebit dimensiones quam data æquatio: e. g. si incognita quantitas, quæ sit radix resultantis æquationis, sit α ; tum

$$2n \times \frac{2n-1}{2} \times \frac{2n-2}{3} \dots \frac{n+1}{n} \text{ habet dimensiones; si modo sit } a, \text{ \& eundem}$$

eundem habebit dimensionum numerum, & sic deinceps. Sint enim radices datæ æquationis $\pi, \rho, \sigma, \tau, \&c.$ & α erit summa e quibusque n radicibus datæ æquationis, & sic deinceps.

Ex. 5. Sit data æquatio $x^6 + qx^4 - rx^3 + sx^2 - tx + v = 0$, assumatur $x^2 + b = ax + c$; & exinde resultat

$$\begin{aligned} x^6 + 3bx^4 + 3b^2x^2 + b^3 &= a^3x^3 + 3a^2cx^2 + 3ac^2x + c^3 \\ Ax^4 + 2Abx^2 + Ab^2 &= Aa^3x^3 + 2Aa^2cx^2 + 2Ac^2x + Ac^3 \\ Bx^2 + Bb &= Bax + Bc. \end{aligned}$$

Fiant coefficientes resultantis æquationis eadem ac datæ æquationis coefficientes, i. e. $3b + A = q$, & $a^3 = r$, &c. & resultabunt quinque æquationes quinque incognitas quantitates habentes; sed hæ æquationes sunt dependentes; sint enim radices datæ æquationis $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$; & $a = \alpha + \beta = \sqrt{r}$: in quo casu hoc problema resolutionem ex hac methodo accipit.

Si generalis quærat solutio, cavendum est, ne ex æquatis terminis datæ & resultantis æquationis ullæ oriantur æquationes ex aliis pendentes; & semper quidem tot vel plures esse debent incognitæ quantitates quot resultantes æquationes.

Si datæ æquationis coefficientes r, s, t , & per ipsos radicem $\alpha, \beta, \gamma, \delta$, &c. terminos exprimantur, & observentur etiam assumptæ æquationis & reducentis coefficientes, inveniri possunt per functiones radicem datæ æquationis incognitæ quantitates in æquatione assumptâ.

Hæc methodus igitur plerumque facile converti potest in transformationem æquationum.

P R O B. XX.

Invenire resolutionem æquationum ex earum transformatione in alias, quarum radices datam habeant relationem ad radices datæ æquationis.

Transformentur datæ æquationes in alias, quarum radices per notas regulas inveniri possint; & talis sit relatio inter radices datæ & transformatæ æquationis, ut e radicibus transformatæ æquationis datis inveniri possint radices datæ æquationis.

N

Ex.

Ex. 1. Sit æquatio cubica $x^3 + qx - r = 0$, fingatur $z - \frac{q}{3z} = x$,
& transformetur cubica æquatio in alteram, cujus radix fit z , & æqua-
tio resultans erit $z^6 - rz^3 - \frac{q^3}{27} = 0$.

Hujus æquationis radices erunt $\sqrt[3]{\frac{1}{2}r + \sqrt{\frac{q^3}{27} + \frac{r^2}{4}}}$; & exinde
radices datæ æquationis erunt $x =$

$$\sqrt[3]{\frac{1}{2}r + \sqrt{\frac{q^3}{27} + \frac{r^2}{4}}} + \frac{\frac{1}{3}q}{\sqrt[3]{\frac{1}{2}r + \sqrt{\frac{q^3}{27} + \frac{r^2}{4}}}} = \sqrt[3]{\frac{1}{2}r + \sqrt{\frac{q^3}{27} + \frac{r^2}{4}}} \\ + \sqrt[3]{\frac{1}{2}r - \sqrt{\frac{q^3}{27} + \frac{r^2}{4}}}.$$

Cor. 1. Sit $a + b\sqrt{-3} = \sqrt[3]{\frac{1}{2}r + \sqrt{\frac{q^3}{27} + \frac{r^2}{4}}}$, & tres
radices cubicæ æquationis $x^3 + qx - r = 0$, erunt re-
spective $a + b\sqrt{-3} + \frac{\frac{1}{3}q}{a + b\sqrt{-3}} = 2a, \frac{-1 + \sqrt{-3}}{2} \times a + b\sqrt{-3} \\ + \frac{\frac{1}{3}q}{\frac{-1 + \sqrt{-3}}{2} \times a + b\sqrt{-3}} \left(\frac{-1 - \sqrt{-3}}{2} a - b\sqrt{-3} \right) = -a - 3b,$
& $\frac{-1 - \sqrt{-3}}{2} \times a + b\sqrt{-3} + \frac{\frac{1}{3}q}{\frac{-1 - \sqrt{-3}}{2} \times a + b\sqrt{-3}} \left(\frac{-1 + \sqrt{-3}}{2} a + b\sqrt{-3} \right) = -a + 3b.$

Sit $a + b = \sqrt[3]{\frac{1}{2}r + \sqrt{\frac{q^3}{27} + \frac{r^2}{4}}}$, & $a - b = \sqrt[3]{\frac{1}{2}r - \sqrt{\frac{q^3}{27} + \frac{r^2}{4}}}$
& tres radices æquationis inveniuntur præcedenti methodo respective
 $2a, -a + b\sqrt{-3},$ & $-a - b\sqrt{-3}.$

Cor. 2.

Cor. 2. Sit æquatio prædicta $x^3 + qx - r = 0$, cujus radices sint α, β, γ ; & sex radices æquationis reducentis, cujus radix est z , erunt respective $\frac{1}{2}\alpha + \beta \sqrt{-\frac{1}{3} + \frac{1}{2}\alpha}$, $-\frac{1}{2}\alpha + \beta \sqrt{-\frac{1}{3} + \frac{1}{2}\alpha}$; $\frac{1}{2}\beta + \alpha \sqrt{-\frac{1}{3} + \frac{1}{2}\beta}$, $-\frac{1}{2}\beta + \alpha \sqrt{-\frac{1}{3} + \frac{1}{2}\beta}$; $\frac{1}{2}\alpha - \frac{1}{2}\beta \sqrt{-\frac{1}{3} - \frac{1}{2}\alpha - \frac{1}{2}\beta}$, $\frac{1}{2}\beta - \frac{1}{2}\alpha \sqrt{-\frac{1}{3} - \frac{1}{2}\alpha - \frac{1}{2}\beta}$: & ex his radicibus inter se comparatis facile constat æquationem esse hujusce formulæ $z^6 + az^3 + b = 0$.

Cor. 3. Sint $\pi, \rho, \sigma, \tau, \upsilon, \phi$ sex prædictæ radices æquationis reducentis $z^6 - rz^3 - \frac{q^3}{27} = 0$ respective; & tres radices datæ æquationis $x^3 + qx - r = 0$ erunt respective $\pi + \rho, \sigma + \tau, \upsilon + \phi$.

Omnia hæc etiam applicari possunt in æquationem cubicam $x^3 - px^2 + qx - r = 0$.

Cor. 4. Ita transformare æquationem $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$ in alteram, ut e resultanti æquatione destruantur (i. e. nihilo fiant æquales) quicunque (m) termini.

Assumatur æquatio exprimens relationem inter radices datæ & reducentis æquationis, & quæ saltem tot involvat incognitas quantitates, quot (m) termini æquationis reducentis destruendi sunt; & resultant tot æquationes, quot termini destruendi sunt.

Ex his æquationibus capite subsequenti inveniri possunt dimensiones æquationis resultantis; si modo ita transformentur hæ (m) æquationes in unam, ut exterminentur ($m-1$) incognitæ quantitates.

Ex. 1. Sit æquatio $x^n + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$, hanc æquationem in alteram n dimensiones habentem transformare, viz. $v^n + Rv^{n-3} + \&c. = 0$, & cujus duo termini, v^{n-1}, v^{n-2} deficient.

Assumo quantitatem ($x^2 + ax + b = 0$) duas incognitas quantitates (a & b) habentem: inveniatur æquatio, cujus radix est v , fiant

termini v^{n-1} & v^{n-2} nihilo æquales; & invenientur $b = \frac{2q}{n}$ & $qa^2 -$

$3ra - \frac{n-2}{n} \times q^2 + 2s = 0$: e quibus æquationibus inveniri possunt

b & a , & exinde per problema quintum æquatio, cujus radix est v .

Sit æquatio $x^3 + qx - r = 0$, affumo æquationem $x^2 + ax + \frac{2q}{3} = v$,
 ubi $qa^2 - 3ra - \frac{q^2}{3} = 0$, & resultat æquatio $v^3 - ra^3 - \frac{2}{3}q^2a^2 - rqa$
 $- r^3 - \frac{2}{27}q^3 = 0$.

Sint $\alpha, \beta, -\alpha - \beta$ tres radices datæ æquationis, & erit

$$a = \frac{2\alpha^3 - 2\beta^3 + 3\alpha^2\beta - 3\beta^2\alpha \times \sqrt{-1} + \sqrt{3} \times \alpha\beta \times \alpha + \beta}{2\alpha^2 + \alpha\beta + \beta^2 \times \sqrt{3}}; = 0,$$

quâ pro valore in æquatione $x^2 + a^2 + b = v$ substitutâ, resultant
 diversæ radices quantitatis (v).

Ex. 2. Sit æquatio $x^n + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + vx^{n-6} -$
 $\&c. = 0$, transformare hanc æquationem in alteram n dimensiones
 habentem $v^n - qv^{n-2} + sv^{n-4} + \&c. = 0$, ubi termini v^{n-1} & v^{n-3} deficiunt.

Affumo æquationem $(x^2 + ax + \frac{2q}{n} = v)$, ubi $ra^3 + 2 \times \frac{n-2}{n} q^2 - 4s$
 $\times a^2 + 5t - \frac{5n-12}{n} qr \times a - \frac{n-4 \times n-2}{3 \cdot n \cdot n} \times 2q^3 + \frac{n-4}{n} \times 2qs$
 $+ r^3 - 2v = 0$; inveniatur æquatio, cujus radix est v , & erit æqua-
 tio quæsitâ.

Sit æquatio $x^4 + qx^2 - rx + s = 0$, & assumatur $x^2 + ax + \frac{1q}{2} = v$,
 ubi $ra^3 + q^2 - 4sa^2 - 2qra + r^3 = 0$; inveniatur æquatio, cujus ra-
 dix sit v ; & æquatio resultans erit biquadratica, formulæ vero qua-
 draticæ,

$$v^4 + qa^2 - 3ra + 2s - \frac{1}{2}q^2 \times v^2 + sa^4 + \frac{qr}{2}a^3 + \frac{q^3}{4} - sq \times a^2 + sr - \frac{rq^2}{4}$$

$$\times a + s^3 + \frac{r^2q}{2} - \frac{q^2s}{2} + \frac{q^4}{16} = 0;$$

cujus ope inveniatur v ; & consequenter ex assumpta æquatione $x^2 +$
 $ax + \frac{q}{2} = v$ inveniri possit (x) radix datæ æquationis quæsitâ.

Quæ

Quæ autem functiones radicum datæ æquationis sint quantitates a & v ; facile constabit pro literis q, r, s substituendo earum valores in terminis radicum datæ æquationis.

Cor. 5. Transformare per hanc methodum æquationem $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + \&c. = 0$, ita ut resultet æquatio hujusce formulæ $v^n + Q = 0$, exigit æquationem reducentem $(1 \times 2 \times 3 \times 4 \dots n-1)$ dimensionum.

Cor. 6. Sint incognitæ in assumptâ quantitate coefficientes plures quam termini tollendi.

Inveniantur termini tollendi, nihilo respectu fiant æquales, & transformentur resultantes æquationes in unam, ita ut exterminentur quædam incognitæ coefficientes, & resultat æquatio duas vel plures incognitas quantitates habens; inveniantur ex singulis his quæcunque radices inter se correspondentes; quibus substitutis, tolluntur ut oportet termini.

Eadem est transformandi methodus, cum requiritur, ut termini respectivi resultantis æquationis quamcunque habeant inter se relationem.

P R O B. XXI.

Invenire æquationes, quas deprimere licet.

Assumatur æquatio (n) dimensionum, & per problema quintum inveniantur aliæ, quarum radices sunt quæcunque datæ functiones assumptæ æquationis radicum; & æquationes resultantes, si modo plures habeant dimensiones quam assumpta æquatio, deprimi possunt in assumptam æquationem.

Cor. Assumatur æquatio $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, cujus radices sint $\alpha, \beta, \gamma, \delta, \&c.$ inveniat æquatio $v^{n \times \frac{n-1}{2}} - P v^{n \times \frac{n-1}{2}-1} + \&c. = 0$, cujus radices sint $\alpha + \beta, \alpha + \gamma, \beta + \gamma, \alpha + \delta, \beta + \delta, \gamma + \delta, \&c.$ & hæc æquatio $(n \times \frac{n-1}{2})$ dimensionum facile deprimi potest in prædictam assumptam æquationem (n) dimensionum.

Æquatio

Æquatio enim resultans $(n \times \frac{n-1}{2})$ dimensionum habet solummodo (n) incognitas quantitates, quæ sunt coefficientes assumptæ æquationis; ex eâ vero resultant $n \times \frac{n-1}{2}$ æquationes (n) prædictas incognitas quantitates solummodo involventes; ergo quoniam plures resultant æquationes quam incognitæ quantitates, e methodo communes divisores inveniendi, constabunt prædictæ incognitæ quantitates.

Si vero radices resultantis æquationis sint $\alpha + \beta + \gamma$, $\alpha + \beta + \delta$, $\alpha + \gamma + \delta$, $\beta + \gamma + \delta$, &c. vel $\alpha + \beta + \gamma + \delta$, &c. tum resultantes æquationes $(n \times \frac{n-1}{2} \times \frac{n-2}{3}, n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}, \&c.)$ dimensionum facile deprimi possunt in æquationes n dimensionum.

Cor. 2. Datis radicibus datæ æquationis (n) dimensionum $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, exinde inveniri possunt radices infinitarum aliarum æquationum.

Transformetur enim hæc æquatio in quolibet alias, quarum radices sint quæcunque functiones radicum datæ æquationis, & constant radices resultantium æquationum e radicibus datæ æquationis.

Sit functio $\frac{ax^m + bx^{m-1} + cx^{m-2} + \&c.}{Ax^r + Bx^{r-1} + Cx^{r-2} + \&c.} = z$ radici resultantis æquationis; ubi m & r sint integri numeri; & $a, b, c, \&c. A, B, C, \&c.$ sint rationales quantitates; & resultans æquatio haud plures habet dimensiones quam data æquatio.

Ex. Sit æquatio $y^n + b = 0$, assumatur $\overline{x+b} \times y = x + a$, exterminetur (y) , & resultat $\overline{1+b} \times x^n + n \times \overline{a+bb} \cdot x^{n-1} + n \times \frac{n-1}{2} \overline{a^2+bb^2} x^{n-2} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \overline{a^3+bb^3} x^{n-3} + \&c. + a^n + bb^n = 0$: sit $bb+a = 0$, & resultat $(A) x^n - n \times \frac{n-1}{2} \overline{ab} x^{n-2} - n \times \frac{n-1}{2} \times \frac{n-2}{3} \overline{ab} \times \overline{a+b}$

x^{n-3}

$x^{n-3} - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} ab \times \overline{a^2 + ab + b^2} x^{n-4} - \&c. = 0$, cujus

æquationis radices erunt $\sqrt[n]{a^{n-1}b} + \sqrt[n]{a^{n-2}b^2} + \sqrt[n]{a^{n-3}b^3} \dots \sqrt[n]{ab^{n-1}}$.

Sit data æquatio $x^n + qx^{n-2} - rx^{n-3} + \&c. = 0$; & terminis datæ & æquationis (A) inter se æquales esse suppositis, resultat æquatio

$$a^2 + \frac{3r}{n-2 \cdot q} - \frac{2q}{n \cdot n-1} = 0.$$

P R O B. XXII.

Datâ æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, & alterâ $z^m - Pz^{m-1} + Qz^{m-2} - Rz^{m-3} + Sz^{m-4} - \&c. = 0$; ubi P, Q, R, S, &c. sint quæcunque functiones coefficientium p, q, r, s, &c. invenire quæ functio radicis (x) sit quantitas (z).

Sint $\alpha, \beta, \gamma, \delta, \&c.$ radices æquationis $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$; pro p, q, r, s, &c. scribantur in coefficientibus alterius æquationis earum valores in terminis radicum $\alpha, \beta, \gamma, \delta, \&c.$ i. e. pro p ejus valor $\alpha + \beta + \gamma + \delta + \&c.$ & invenitur quæ functio radicum $\alpha, \beta, \gamma, \delta, \&c.$ sint coefficientes P, Q, R, S, &c. inveniatur radix æquationis resultantis, & fit problema.

Hujus problematis exempla antea tradidi.

P R O B. XXIII.

Datâ quantitate utcunque irrationali; invenire alteram, quæ in datam multiplicata, rationalem faciat quantitatem.

Data irrationalis quantitatis inveniantur singuli diversi valores, quibus in sese continuo ductis, contentum erit rationalis quantitas; dividatur hoc contentum per datam quantitatem, & quotiens erit irrationalis quantitas quæsitâ.

Ex. I. Invenire irrationalem quantitatem, quæ in datam $\sqrt{a+bx} + \sqrt[3]{c+dx+ex^2} + f + gx$ multiplicata, rationalem quantitatem facit.

Diversi

Diverſi ſex valores datæ irrationalis quantitatis $\sqrt[3]{a+bx} + \sqrt[3]{c+dx+ex^2} + f+gx$ erunt reſpective

$$\sqrt[3]{a+bx} + \sqrt[3]{c+dx+ex^2} + f+gx,$$

$$\& - \sqrt[3]{a+bx} + \sqrt[3]{c+dx+ex^2} + f+gx,$$

$$\& + \sqrt[3]{a+bx} - \frac{1+\sqrt{-3}}{2} \sqrt[3]{c+dx+ex^2} + f+gx,$$

$$\& - \sqrt[3]{a+bx} - \frac{1+\sqrt{-3}}{2} \sqrt[3]{c+dx+ex^2} + f+gx,$$

$$\& + \sqrt[3]{a+bx} - \frac{1-\sqrt{-3}}{2} \sqrt[3]{c+dx+ex^2} + f+gx,$$

$$\& - \sqrt[3]{a+bx} - \frac{1-\sqrt{-3}}{2} \sqrt[3]{c+dx+ex^2} + f+gx;$$

contentum ex his ſex valoribus continuo in ſeſe ductis erit rationalis quantitas $f+gx^6 - 3f+gx^4 \times a+bx + 2f+gx^3 \times c+dx+ex^2 + 3f+gx^2 \times a+bx^2 + 6f+gx \times a+bx \times c+dx+ex^2 + c+dx+ex^2^2 - a+bx^3$: & contentum ex omnibus his præter primam irrationali quantitatē (quæ ſit data), erit irrationalis quantitas quæſita.

Cor. 1. Hoc problemate irrationalis quantitas (*A*) a denominatore in numeratorem, & vice verſa transferri poteſt.

Inveniatur rationalis quantitas *B*, cujus data irrationalis eſt divisor; ſi vero $C = \frac{B}{A}$, & pro *A* ſubſtituatur $\frac{B}{A}$; & id, quod requirebatur, perficitur.

Cor. 2. Datâ æquatione irrationales & incognitam quantitatem *x* involvente, & facile inveniri poſſint dimenſiones æquationis reſultantis; ſi modo ita reducatür æquatio data, ut exterminentur irrationales quantitates.

Ducantur maximæ dimenſiones incognitæ quantitatis (*x*) in datâ æquatione

æquatione contentæ in contentum e singulis dimensionibus (quarum radices in datâ æquatione continentur) in se invicem multiplicatis, & rectangulum exinde resultans fit A .

Ducantur minimæ dimensiones incognitæ quantitatis (x) in datâ æquatione contentæ in prædictum contentum e singulis dimensionibus (quarum radices in datâ æquatione continentur) in se invicem multiplicatis, & dicatur resultans rectangulum B ; & dimensiones quæsitæ erunt $A+B$, si modo B sit negativa quantitas; si vero B sit affirmativa quantitas; $A-B$; erunt enim hoc in casu valores (B) nihilo æquales.

Cum vero terminorum, in quibus inveniuntur maximæ vel minimæ dimensiones incognitæ quantitatis (x), quidam valores evanescant; tum evanescant etiam quædam dimensiones incognitæ quantitatis (x) in æquatione resultanti: quot autem evanescant dimensiones incognitæ quantitatis (x) in resultanti æquatione e terminis, in quibus maximæ vel minimæ dimensiones incognitæ quantitatis (x) inveniuntur, & eorum proximis, &c. facile detegi possint.

Ex. 1. Sit data æquatio $\sqrt[2]{a+bx+cx^2} + \sqrt[3]{f+gx+bx^2+kx^3} + \sqrt[5]{p+qx+rx^2+sx^3} + P + Qx = 0$. In hac æquatione dimensio maximæ incognitæ quantitatis in primâ irrationali quantitate $\sqrt[2]{a+bx+cx^2}$ erit $\frac{2}{2} = 1$; in secundâ irrationali quantitate $\sqrt[3]{f+gx+bx^2+kx^3}$ dimensio, etiam erit $\frac{3}{3} = 1$; in tertiâ $\sqrt[5]{p+qx+rx^2+sx^3}$ erunt dimensiones $3 \times \frac{2}{5} = \frac{6}{5}$; in rationali quantitate $P + Qx$ dimensio erit 1; dimensiones igitur termini, in quo dimensiones incognitæ quantitatis inveniuntur maximæ in datâ æquatione, erunt $\frac{6}{5}$: ducantur hæ dimensiones ($\frac{6}{5}$), in contentum ex omnibus

bus dimensionibus ($2 \cdot 3 \cdot 5$), quarum radices continentur in datâ æquatione, & fit rectangulum $2 \cdot 3 \cdot 5 \cdot \frac{6}{5} = 36$; ergo dimensiones incognitæ quantitatis (x) in æquatione resultanti erit 36; ni terminorum, in quibus maximæ vel minimæ inveniuntur dimensiones incognitæ quantitatis (x), quidam valor vel valores evanescant; in hoc exemplo unus solummodo terminus $\sqrt[5]{5x^3 + \&c.}$ involvitur in datâ æquatione, in quo maximæ $\left(\frac{6}{5}\right)$ continentur dimensiones: ergo nullus datur evanescens valor incognitæ quantitatis (x) e terminis, in quibus maximæ continentur dimensiones: termini autem, in quibus minimæ vel potius nullæ dimensiones inveniuntur incognitæ quantitatis (x) erunt $\sqrt[2]{a}$, $\sqrt[3]{f}$, $\sqrt[5]{p}$, P ; si vero $\pm \sqrt[2]{a} + \sqrt[3]{f} + \sqrt[5]{p} + P = 0$; tum nihilo erit æqualis valor incognitæ quantitatis (x), sin aliter vero non. Si vero evanescat unus valor incognitæ quantitatis (x), tum inquirendum est, utrum duo, tres, &c. termini resultantis æquationis nihilo sint æquales, necne.

Ex. 2. Sit data æquatio $\sqrt[5]{8x^2 + 2x + \frac{7}{3x}} + \sqrt{64x^4 - 20} + \sqrt[6]{10x} + \sqrt[2]{64x^4 - 20} = 0$; invenire, quot dimensiones habet data æquatio e surdis quantitibus liberata.

In hac æquatione maximæ dimensiones incognitæ quantitatis in priori irrationali quantitate erunt $\frac{2}{5}$ vel $\frac{4}{2 \cdot 5} = \frac{2}{5}$; maximæ vero dimensiones in posteriori quantitate erunt $\frac{4}{2 \cdot 6} = \frac{1}{3}$: ergo dimensiones termini, in quo dimensiones incognitæ quantitatis (x) inveniuntur maximæ,

maximæ, erunt $\frac{2}{5}$: ducantur hæ dimensiones $\frac{2}{5}$ in contentum e singulis dimensionibus, quarum radices in datâ æquatione continentur, & resultabunt $\frac{2}{5} \times 5 \times 6 \times 2 = 24 = A$.

Minimæ autem dimensiones incognitæ quantitatis x in priori irrationali quantitate erunt $-\frac{1}{5}$, in posteriori vero erunt nihilo æquales; ergo dimensiones termini, in quibus minimæ inveniuntur dimensiones incognitæ quantitatis erunt $-\frac{1}{5}$, & exinde $-\frac{1}{5} \times 5 \times 6 \times 2 = -12 = B$: & $A + B = 24 + 12 = 36$. Sed quoniam evanescent

1×6 valores terminorum $\sqrt{8x^2} = \sqrt{64x^4}$ &c. in quibus maximæ inventæ fuere dimensiones incognitæ quantitatis x , & inter se æquales; æquatio resultans e furdis libera haud dimensiones 36 habere potest; habet vero 30 dimensiones.

Alia etiam deduci potest confimilis methodus transformandi æquationes, ita ut irrationales quantitates exterminentur; etiamque numerum dimensionum cujuscunque incognitæ quantitatis (x) in æquatione resultanti deducendi.

Per seriem infinitam terminorum juxta dimensiones incognitæ quantitatis (x) progredientium, i. e. vel ascendendum vel descendendum inveniuntur singuli diversi valores ex irrationalibus quantitatibus: ducantur hæ quantitates in sese, & exinde resultabit æquatio quæsitæ, & quæ nullas habet irrationales quantitates.

Ex. Sit æquatio $\sqrt{16x^4 + 100} + \sqrt{9x^2 + 36} + 7 + 2x = 0$: quatuor diversi valores harum irrationalium quantitatum per infinitam seriem terminorum juxta dimensiones incognitæ quantitatis x progredientium i. e. descendendum, invenientur respective

$$4x^2 + 2 + 3x + 7 + \frac{6}{x} \text{ \&c. ad terminum } \frac{\pi}{x^6} = 0,$$

$$4x^2 + \overline{2-3}x + 7 - \frac{6}{x} \&c. \text{ ad prædictum terminum } = 0,$$

$$-4x^2 + \overline{2+3}x + 7 + \frac{6}{x} \&c. = 0,$$

$$-4x^2 + \overline{2-3}x + 7 - \frac{6}{x} \&c. = 0;$$

ducantur hi diverſi valores in ſeſe, & reſultabit æquatio quæſita
 $256x^8 - 416x^6 - 896x^5 \&c. = 0.$

Facile ex hac reſolutionis methodo conſtant diſenſiones, ad quas aſſurget incognita quantitas (x) in æquatione reſultanti a ſurdis liberâ.

P R O B. XXIV.

Quantitate datâ, quæ ſimplicium terminorum juxta diſenſiones cujuſlibet literæ (x) progredientium ſeriem ($a+b+c+d+e+f+g+h+k+\&c.$) exprimit, invenire quantitatem alternorum ſeriei terminorum ($a+c+e+g+k+\&c.$) ſummæ æqualem; etiamque quantitates terminorum ſeriei propoſitæ binis vel ternis vel denique n intervallis a ſe invicem diſtantium ſummis reſpectivæ æquales.

Sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ reſpectivæ radices $(n+1)$ poteſtatis unitatis; i. e. ſit $n+1=2$, & α, β erunt reſpectivæ 1 & -1 ; ſit $n+1=3$, & α, β, γ erunt reſpectivæ $1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}$.

In datâ quantitate pro literâ (x) ſubſtituantur reſpectivæ $\alpha x, \beta x, \gamma x, \delta x, \epsilon x, \&c.$ quantitatum reſultantium ſumma per numerum $(n+1)$ diviſa erit quæſita quantitas.

Ex. 1. Sit quantitas $\overline{a+x^m}$, quæ exprimit ſeriem ſequentem ſimplicium terminorum juxta diſenſiones literæ (x) progredientium $a^m + ma^{m-1}x + m \cdot \frac{m-1}{2} a^{m-2}x^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}x^3 + \&c.$ invenire quantitatem alternis ſeriei terminis æqualem.

In

In hoc exemplo $n + 1 = 2$; & quoniam radices quadraticæ unitatis sunt 1 & -1 , pro x in datâ quantitate scribantur respective $+x$ & $-x$; & resultant duæ quantitates $a + x^m$, & $a - x^m$; quarum summa per numerum $n + 1 = 2$ divisa, erit quæsitâ quantitas, i. e.

$$\frac{a + x^m + a - x^m}{2} = a^m + m \times \frac{m-1}{2} a^{m-2} x^2 + \&c.$$

Ex. 2. Eâdem quantitate datâ, invenire quantitates primi, quarti, septimi, decimi, & terminorum summæ æqualem. In hoc exemplo

$n + 1 = 3$, & cubicæ radices numeri (1), erunt 1, $\frac{-1 + \sqrt{-3}}{2}$,

$\frac{-1 - \sqrt{-3}}{2}$: in datâ igitur quantitate pro (x) substituantur x ,

$\frac{-1 + \sqrt{-3}}{2} x$, $\frac{-1 - \sqrt{-3}}{2} x$ respective; & resultant quantitates, quarum summa per numerum ($n + 1 = 3$) divisa erit, quæsitâ

quantitas, i. e. $\frac{a + x^m + a - \frac{-1 + \sqrt{-3}}{2} x^m + a - \frac{-1 - \sqrt{-3}}{2} x^m}{3} =$

$$a^m + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3} x^3 + \&c.$$

Ex. 3. Datâ quantitate $a + bx + c + dx + ex^{2l}$, exprimente seriem terminorum juxta dimensiones literæ (x) progredientium; invenire quantitatem alternis hujus seriei terminis æqualem. Hoc exemplo $n + 1 = 2$, ergo pro x in datâ quantitate scribantur respective x & $-x$, & resultant quantitates $a + bx + c + dx + ex^{2l}$, & $a - bx + c - dx + ex^{2l}$; quarum summa per numerum ($n + 1 = 2$) divisa, erit quæsitâ quantitas.

Cor. Sint $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \&c.$ respectivæ radices æquationis $x^n - 1 = 0$; & data quantitate (A), quæ simplicium terminorum juxta dimensiones cujuscunque incognitæ quantitatis (x) progredientium summam

summam exprimit: In datâ quantitate A pro incognitâ quantitate (x) scribantur respective (x) $\alpha x, \beta x, \gamma x, \delta x, \epsilon x, \zeta x, \&c.$ & sint quantitates resultantes respective $A, B, C, D, E, F, \&c.$ tum summa primi termini, & terminorum, quorum distantiae a primo sint respective $n, 2n, 3n, 4n, 5n, \&c.$ erit per problema $\frac{A+B+C+D+E+F+\&c.}{n}$.

Summa autem secundi termini, & terminorum, quorum distantiae a secundo sint respective $n, 2n, 3n, 4n, 5n, \&c.$ erit

$$\frac{\alpha^{n-1}A + \beta^{n-1}B + \gamma^{n-1}C + \delta^{n-1}D + \epsilon^{n-1}E + \zeta^{n-1}F + \&c.}{n}$$

Summa autem tertii termini, & terminorum, quorum distantiae a tertio sint respective $n, 2n, 3n, 4n, 5n, \&c.$ erit

$$\frac{\alpha^{n-2}A + \beta^{n-2}B + \gamma^{n-2}C + \delta^{n-2}D + \epsilon^{n-2}E + \zeta^{n-2}F + \&c.}{n}$$

Summa denique termini, cujus distantia a primo sit r , & terminorum, quorum distantiae a prædicto termino sint respective $n, 2n, 3n, 4n, 5n, \&c.$ erit $\frac{\alpha^{n-r}A + \beta^{n-r}B + \gamma^{n-r}C + \delta^{n-r}D + \epsilon^{n-r}E + \zeta^{n-r}F + \&c.}{n}$.

Ex. 1. Sit data quantitas $\overline{a+x^m} = a^m + ma^{m-1}x + \&c.$ & quantitas primo, tertio, quinto, septimo, &c. terminis æqualis erit per problema $\frac{\overline{a+x^m} + \overline{a-x^m}}{2}$; hoc enim in casu $n=2$; & exinde radices (α, β) æquationis $x^2 - 1 = 0$, erunt 1 & -1 ; substituantur $+x$ & $-x$ pro x & resultant quantitates A & B respective $\overline{a+x^m}$ & $\overline{a-x^m}$.

Quantitas autem æqualis secundo, quarto, sexto, octavo, &c. terminis erit $\frac{\overline{a+x^m} - \overline{a-x^m}}{2}$; erit enim hæc summa $\frac{\alpha^{2-1}(\alpha)A + \beta^{2-1}(\beta)B}{2}$;

sed α & β sunt 1 & -1 , A & B sunt $\overline{a+x^m}$ & $\overline{a-x^m}$; ergo summa quæ sita erit $\frac{\overline{a+x^m} - \overline{a-x^m}}{2}$.

PROB.

PROB. XXV.

Invenire formulas quantitatum, quibus in se invicem multiplicatis, orientur quæcunque datæ quantitates.

Per problema 2 3 constat, cujus generis vel formulæ sint irrationales quantitates, quibus in sese multiplicatis rationale contentum facit; & exinde facile constabit, cujus generis vel formulæ sint irrationales quantitates, quibus in sese ductis irrationalem cujuscunque generis quantitatem facit.

His igitur cognitis, assumantur quantitates talis formulæ, qualem exigant datæ quantitates; ducantur hæ quantitates in se invicem, & resultat quæsitæ formulæ quantitas: æquatis igitur datæ & resultantis quantitatis terminis, invenientur æquationes, e quibus erui possint quantitates quæsitæ.

In genere ita assumantur quantitates, ut tot vel plures sint incognitæ quantitates, quam æquationes independentes resultantes ex æqualibus terminis.

Ex. 1. Data quantitas $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + \&c.$ oriri potest e multiplicatione terminorum $x - \alpha$, $x - \beta$, $x - \gamma$, $x - \delta$, $x - \epsilon$, &c. quorum numerus est n ; hi enim termini in se invicem multiplicati producant quantitatem dati generis $x^n - \alpha + \beta + \gamma + \delta + \&c. x^{n-1} + \alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \gamma\delta + \&c. x^{n-2} \&c.$ æquatis igitur hujus & datæ quantitatis terminis resultabunt (n) æquationes tot incognitas independentes quantitates habentes, ergo data quantitas $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$ oriri potest e multiplicatione horum factorum $x - \alpha$, $x - \beta$, $x - \gamma$, $x - \delta$, &c.

Ex. 2. Prædicta quantitas $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, oriri potest e multiplicatione factorum $\sqrt{x+A} + x+a$, $-\sqrt{x+A} + x+a$, $\sqrt{x+B} + x+b$, $-\sqrt{x+B} + x+b$, &c. ad n terminos: resultans enim æquatio erit quæsitæ formulæ, & n erunt æquationes (n) incognitas quantitates habentes.

Eodem

Eodem prorsus modo inveniri possint infinitæ constructiones, e quibus orientur quæcunque datæ quantitates.

Eâdem methodo inveniri possit, an e quantitatibus datarum formularum assumptis formari possit data quantitas.

Reducantur quantitates assumptæ, ita ut habeant eandem formulam, quam habeat data quantitas; & terminis datæ & resultantis quantitatis inter se æquatis, sint tot vel plures incognitæ quantitates, quot resultantes independentes æquationes, e quibus æquationibus erui possint incognitæ quantitates quæsitæ.

Ex. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, cujus radices

sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ invenire utrum fractio $\frac{1}{x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c.}$

fit summa e fractionibus hujusce formulæ $\frac{a}{x-\alpha}, \frac{b}{x-\beta}, \frac{c}{x-\gamma}, \frac{d}{x-\delta},$

$\frac{e}{x-\epsilon}, \&c.$ Ex hypothesi $\frac{1}{x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c.} = \frac{a}{x-\alpha} +$

$\frac{b}{x-\beta} + \frac{c}{x-\gamma} + \frac{d}{x-\delta} + \frac{e}{x-\epsilon} + \&c.$ reducantur hæ fractiones

assumptæ in formulam, quam habet data quantitas, i. e. in communem denominatorem, & fit

$$\frac{1}{x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c.} = \frac{a+b+c+d+e+\&c. x^{n-1} + a \times \beta + \gamma + \delta + \&c.}{x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c.} + \frac{b \times \alpha + \gamma + \delta + \&c. + c \times \alpha + \beta + \delta + \&c. + d \times \alpha + \beta + \gamma + \&c. \times x^{n-2} + a \times \beta \gamma + \beta \delta + \gamma \delta + \&c. + b \times \alpha \gamma + \alpha \delta + \gamma \delta + \&c. + \&c. \times x^{n-3}}{x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c.}$$

Et terminis datæ (1) & resultantis numeratoris æquatis, eundem enim habent denominatorem; resultant (n) simplices & independentes æquationes, tot incognitas quantitates ($a, b, c, d, e, \&c.$) habentes, e quibus simplicibus æquationibus facile invenientur incognitæ quantitates

$a =$

$$a = \frac{1}{\alpha - \beta \times \alpha - \gamma \times \alpha - \delta \times \alpha - \epsilon \times \alpha \&c.}, \quad b = \frac{1}{\beta - \alpha \times \beta - \gamma \times \beta - \delta \times \beta - \epsilon \times \beta \&c.},$$

$$c = \frac{1}{\gamma - \alpha \times \gamma - \beta \times \gamma - \delta \times \gamma - \epsilon \times \gamma \&c.}, \quad d = \frac{1}{\delta - \alpha \times \delta - \beta \times \delta - \gamma \times \delta - \epsilon \times \delta \&c.},$$

$$e = \frac{1}{\epsilon - \alpha \times \epsilon - \beta \times \epsilon - \gamma \times \epsilon - \delta \times \epsilon \&c.} \&c.$$

Cor. Hinc inveniri possunt infinitæ diversæ coefficientium cujus-
cunque æquationis constitutiones, quæcunque sit constructio æquatio-
nis, sequitur enim e singulis coefficientibus valor correspondens.

P R O B. XXVI.

Data resolutione, æquationem invenire.

Præcedentis lemmatis vel problematis quinti ope ita reducatur data
æquatio, ut ex eâ exterminetur irrationales quantitates, & fit pro-
blema.

Æquatio resultans sæpe reduci possit in alias minores habentes di-
mensiones: Hoc fit divisores æquationis resultantis inveniendo; si
nullos recipias divisores, tum haud reduci potest resultans æquatio.

Ex. 1. Sit x incognita quantitas, & data resolutio $x =$
 $\sqrt[3]{-\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}} + \sqrt[3]{-\frac{1}{2}b - \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}$; ita reducatur data æ-
quatio, ut ex eâ exterminetur irrationales quantitates, & resultat
æquatio $x^9 + 3bx^6 + 3b^2 + a^3x^3 + b^3 = 0$, cujus cubicæ æquationes
 $x^3 + ax^3 + b = 0$, $x^3 - \frac{1 + \sqrt{-3}}{2}ax + b = 0$, $x^3 - \frac{1 - \sqrt{-3}}{2}ax + b$
 $= 0$ erunt divisores.

Hinc constat, quod resolutio $\sqrt[3]{-\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}} + \sqrt[3]{-\frac{1}{2}b - \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}$
potius dici possit resolutio æquationis novem quam trium dimen-
sionum.

P

Ex. 2.

Ex. 2. Sit resolutio $x = \sqrt[3]{-b + \sqrt{b^2 + q^3}} + \frac{A}{\sqrt[3]{-b + \sqrt{b^2 + q^3}}}$
ita reducatur hæc æquatio, ut exterminentur irrationales quantitates,
& resultat æquatio

$$x^3 - 3Ax + \frac{q^3 + A^3}{q^3}b - b^2 + q^3 \times \frac{A^3 - q^3}{q^3} = 0.$$

Si $A = q$, tum omnis valor resultantis æquationis alterum habet
sibi ipsi æqualem.

Ex. 3. Sit resolutio $x = \sqrt[n]{a + \sqrt{b^2}} + A\sqrt[n]{a - \sqrt{b^2}}$, exterminentur
irrationales quantitates, & resultat æquatio $(n \times n)$ dimensio-
num, cujus æquationes

$$\begin{aligned} x^n - n \times a^{\frac{n-1}{2}} \sqrt{b^2}^{\frac{1}{2}} Ax^{n-2} + n \times \frac{n-3}{2} \times a^{\frac{n-5}{2}} \sqrt{b^2}^{\frac{3}{2}} A^2 x^{n-4} - n \times \frac{n-5}{2} \times \frac{n-7}{3} \times \\ a^{\frac{n-7}{2}} \sqrt{b^2}^{\frac{5}{2}} A^3 x^{n-6} + n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} \times a^{\frac{n-9}{2}} \sqrt{b^2}^{\frac{7}{2}} A^4 x^{n-8} - \&c. - A^n + 1 \\ \times a \pm A^{n-1} \sqrt{b^2} = 0; x^n - n \pi a^{\frac{n-1}{2}} \sqrt{b^2}^{\frac{1}{2}} Ax^{n-2} + n \times \frac{n-3}{2} \pi^2 a^{\frac{n-5}{2}} \sqrt{b^2}^{\frac{3}{2}} A^2 x^{n-4} - \\ n \times \frac{n-4}{2} \times \frac{n-5}{3} \pi^3 \times a^{\frac{n-7}{2}} \sqrt{b^2}^{\frac{5}{2}} A^3 x^{n-6} + \&c. - A^n + 1 \times a \pm A^{n-1} \times \\ \sqrt{b^2} = 0; x^n - n \rho a^{\frac{n-1}{2}} \sqrt{b^2}^{\frac{1}{2}} Ax^{n-2} + n \times \frac{n-3}{2} \rho^2 \times a^{\frac{n-5}{2}} \sqrt{b^2}^{\frac{3}{2}} A^2 x^{n-4} - n \times \\ \frac{n-4}{2} \times \frac{n-5}{3} \rho^3 \times a^{\frac{n-7}{2}} \sqrt{b^2}^{\frac{5}{2}} A^3 x^{n-6} + \&c. - A^n + 1 \times a \pm A^{n-1} \times \sqrt{b^2} = 0, \\ \&c. \& \text{ ubi } 1, \pi, \rho, \&c. \text{ respective diversas radices æquationis } x^n - 1 \\ = 0 \text{ denotent.} \end{aligned}$$

Ex. 4. Sit resolutio $x = \sqrt[n]{a + \sqrt{b^2}} + \frac{A}{\sqrt[n]{a + \sqrt{b^2}}}$; exterminentur
irrationales quantitates, & resultat æquatio

$x^n -$

$$x^n - nAx^{n-2} + n \times \frac{n-3}{2} A^2 x^{n-4} - n \times \frac{n-4}{2} \times \frac{n-5}{3} A^3 x^{n-6} + \&c. +$$

$$\frac{A^n + a^2 - b^2}{a^2 - b^2} \times a - b^2 \times \frac{A^n - a^2 + b^2}{a^2 - b^2}$$

Cor. In utrisque his casibus, cum A fiat 1, tum omnis valor resultantis æquationis alterum habet sibi ipsi æqualem.

Cor. 2. Sit $x^n - 1 = 0$, sint autem ejus quadratici divisores $x^2 - \pi x + 1 = 0$, $x^2 - \rho x + 1 = 0$, $x^2 - \sigma x + 1 = 0$, $x^2 - \tau x + 1 = 0$, quarum æquationum sint duæ radices respective α , β ; & γ , δ ;

& ϵ , ζ ; & η , θ ; &c. i. e. $\beta = \frac{1}{\alpha}$, $\delta = \frac{1}{\gamma}$, $\epsilon = \frac{1}{\zeta}$, $\theta = \frac{1}{\eta}$, &c. & quo-

niam radices e resolutione $x = \sqrt[n]{a + \sqrt{b}} + \frac{A}{\sqrt[n]{a + \sqrt{b}}}$ erunt

$$\sqrt[n]{a + \sqrt{b}} + \frac{A}{\sqrt[n]{a + \sqrt{b}}}, \alpha \sqrt[n]{a + \sqrt{b}} + \frac{A}{\alpha \sqrt[n]{a + \sqrt{b}}}, \beta \sqrt[n]{a + \sqrt{b}}$$

$$+ \frac{A}{\beta \sqrt[n]{a + \sqrt{b}}}, \gamma \sqrt[n]{a + \sqrt{b}} + \frac{A}{\gamma \sqrt[n]{a + \sqrt{b}}}; \&c. \text{ radices e resolu-}$$

tione $\sqrt[n]{a + \sqrt{b}} + A \sqrt[n]{a - \sqrt{b}}$ erunt $\sqrt[n]{a + \sqrt{b}} + A \sqrt[n]{a - \sqrt{b}}$,

$$\alpha \sqrt[n]{a + \sqrt{b}} + \beta A \sqrt[n]{a - \sqrt{b}}, \gamma \sqrt[n]{a + \sqrt{b}} + \delta A \sqrt[n]{a - \sqrt{b}}, \epsilon \sqrt[n]{a + \sqrt{b}}$$

$$+ \zeta A \sqrt[n]{a - \sqrt{b}}, \&c.$$

Ex. 4. Sit resolutio $x = a \sqrt[n]{p} + b \sqrt[n]{p^2} + c \sqrt[n]{p^3} + d \sqrt[n]{p^4} \dots$

$A \sqrt[n]{p^{n-4}} + B \sqrt[n]{p^{n-3}} + C \sqrt[n]{p^{n-2}} + D \sqrt[n]{p^{n-1}}$, exterminentur irrationales quantitates, & resultat æquatio (n) dimensionum $x^n - n$

$$aD + bC + cB + dA + \&c. x^{n-2} + \&c. = 0.$$

1. Sit resolutio $x = a \sqrt[3]{p} + \sqrt[3]{p^2}$, ita reducatur, ut exterminentur irrationales quantitates, & resultat æquatio $x^3 - 3apx - p^2$

— $a^3 p = 0$. Si vero requiratur resolutio cubicæ æquationis $x^3 + qx - r = 0$: assumantur $q = -3ap$, & $r = p^3 + a^3 p$; & consequenter $a = \frac{-q}{3p}$, $p^3 - \frac{q^3 p}{27 p^3} = r$; & exinde $p^3 - rp = \frac{q^3}{27}$; unde $p^3 = \sqrt[3]{\frac{1}{4}r^3 + \frac{q^3}{27}} + \frac{r}{2}$, & constat solutio.

2. Sit resolutio $x = \sqrt[3]{p} + a \sqrt[3]{p^2}$, exterminentur irrationales quantitates, & resultat $x^3 - 3apx - p - a^3 p^2 = 0$, fiat hæc eadem ac cubica æquatio $x^3 + qx - r = 0$, & erit $p^3 - rp - \frac{q^3}{27} = 0$ æquatio, cujus ope invenitur p ; & exinde $a = \frac{-q}{3p}$; & $x = \sqrt[3]{p} + a \sqrt[3]{p^2}$.

3. Sit resolutio $x = \sqrt[n]{p} + a \sqrt[n]{p^{n-1}}$; exterminentur incognitæ quantitates, & æquatio resultans erit $x^n - n \times ap x^{n-2} + n \times \frac{n-3}{2} a^2 p^2 x^{n-4} - n \times \frac{n-4}{2} \times \frac{n-5}{3} a^3 p^3 x^{n-6} + n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} a^4 p^4 x^{n-8} - \&c. - p - a^n p^{n-1} = 0$.

4. Sit resolutio $x = \sqrt[n]{p^m} + a \sqrt[n]{p^{n-m}}$, exterminentur irrationales quantitates, & resultat æquatio $x^n - n p a x^{n-2} + n \times \frac{n-3}{2} p^2 a^2 x^{n-4} - n \times \frac{n-4}{2} \times \frac{n-5}{3} p^3 a^3 x^{n-6} + n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} p^4 a^4 x^{n-8} - \&c. - p^m - a^n p^{n-m} = 0$.

Infinitæ confimiles inveniri possunt æquationes, quarum resolutiones cognoscuntur.

Cor. Hinc inveniri possunt infinitæ æquationes (n) dimensionum, ($n-1$) incognitas quantitates habentes, quarum resolutiones cognoscuntur.

Assumatur enim $x = \sqrt[n]{p} + a \sqrt[n]{p^2} + b \sqrt[n]{p^3} \dots k \sqrt[n]{p^{n-2}} + l \sqrt[n]{p^{n-1}}$; hæc

hæc autem resolutio (n) habet incognitas quantitates: substituatur pro unâ literâ quæcunque quantitas; & exterminentur omnes irrationales quantitates, & resultat æquatio (n) dimensionum ($n-1$) incognitas quantitates habens.

Sit data æquatio $x^n + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + \&c. = 0$. Fiant correspondentes datæ & resultantis æquationis coefficientes inter se æquales, & (n) resultant æquationes ($n-1$) solummodo incognitas quantitates habentes, e methodo communes divisores inveniendi erui possunt incognitarum quantitarum valores, si modo quosdam admittant; etiamque, si modo ita reducantur (n) resultantes æquationes, ut ex eâ exterminentur omnes incognitæ quantitates, constat, quo casu resolutionem hujusmodi admittet data æquatio.

Cor. 2. Sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ radices æquationis (n) dimensionum, cujus resolutio requiritur: sint etiam $\pi, \rho, \sigma, \tau, \&c.$ radices æquationis $x^n - 1 = 0$; & $p = 1$; supponantur

$$\alpha = \pi a + \pi^2 b + \pi^3 c + \pi^4 d + \pi^5 e + \&c.$$

$$\beta = \rho a + \rho^2 b + \rho^3 c + \rho^4 d + \rho^5 e + \&c.$$

$$\gamma = \sigma a + \sigma^2 b + \sigma^3 c + \sigma^4 d + \sigma^5 e + \&c.$$

$$\delta = \tau a + \tau^2 b + \tau^3 c + \tau^4 d + \tau^5 e + \&c.$$

$$\&c. = \&c.$$

datis autem (n) quantitatibus $\alpha, \beta, \gamma, \delta, \&c.$ & inventis (n) valoribus $\pi, \rho, \sigma, \tau, \&c.$ e (n) fictitiis simplicibus æquationibus inveniantur valores quantitarum $a, b, c, d, e, \&c.$ quibus cognitis resolutio quæsitæ

$$\text{erit } x = a \sqrt[n]{p} + b \sqrt[n]{p^2} + c \sqrt[n]{p^3} + d \sqrt[n]{p^4} + e \sqrt[n]{p^5} + \&c.$$

Ex hoc problemate inveniri possunt quilibet æquationum numerus, quarum dantur resolutiones.

Assumatur resolutio, & transformetur hæc æquatio, ita ut deleantur irrationales quantitates, & resultat æquatio, cujus datur resolutio.

Et sic in quam plurimis casibus resolvi potest data æquatio.

Ex. 1. Sit data æquatio $x^3 + qx - r = 0$; assumatur $x = \sqrt[3]{\alpha} + \sqrt[3]{\beta}$,
& consequenter $x^3 = \alpha + \beta + 3 \sqrt[3]{\alpha\beta} \times \sqrt[3]{\alpha} + \sqrt[3]{\beta}$

$$qx =$$

$qx = \sqrt[3]{a} + \sqrt[3]{\beta}$
 $-r = -r$. Supponatur $\alpha + \beta - r = 0$, & $3\sqrt[3]{\alpha\beta}$
 $+ q = 0$; unde $\alpha + \beta = r$, & $\alpha\beta = \frac{-q^3}{27}$; & α & β erunt duæ ra-
 dices quadraticæ æquationis $z^2 - rz + \frac{q^3}{27} = 0$.

Tres radices datæ æquationis erunt respective $\sqrt[3]{\alpha} + \sqrt[3]{\beta}$, $-\frac{1+\sqrt{-3}}{2}\sqrt[3]{\alpha} - \frac{1-\sqrt{-3}}{2}\sqrt[3]{\beta}$, $-\frac{1-\sqrt{-3}}{2}\sqrt[3]{\alpha} - \frac{1+\sqrt{-3}}{2}\sqrt[3]{\beta}$; & æquatio, cujus radices sint prædictæ quantitates, erit $x^3 - 3\sqrt[3]{\alpha\beta}x - \alpha - \beta = 0$.

Ex. 2. Sit data æquatio $x^4 + qx^2 - rx + s = 0$; assumatur pro
 ejus resolutione $x = \sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma}$; & resultat

$$\begin{aligned} x^4 &= \alpha^2 + \beta^2 + \gamma^2 + 4 \times \alpha + \beta + \gamma \times \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} + \alpha^{\frac{1}{2}}\gamma^{\frac{1}{2}} + \beta^{\frac{1}{2}}\gamma^{\frac{1}{2}} + 8 \times \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}} \times \alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}} + \gamma^{\frac{1}{2}} + \\ &\quad 6 \times \alpha\beta + \alpha\gamma + \beta\gamma \\ + qx^2 &= 2q \times \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} + \alpha^{\frac{1}{2}}\gamma^{\frac{1}{2}} + \beta^{\frac{1}{2}}\gamma^{\frac{1}{2}} + q \times \alpha + \beta + \gamma \\ - rx &= -r \times \alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}} + \gamma^{\frac{1}{2}} \\ + s &= s. \end{aligned}$$

Supponantur $4 \times \alpha + \beta + \gamma + 2q = 0$; $\alpha^2 + \beta^2 + \gamma^2 + 6 \times \alpha\beta + \alpha\gamma + \beta\gamma$
 $+ q \times \alpha + \beta + \gamma + s = 0$; $8 \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}} - r = 0$; e primâ æquatione
 colligitur $\alpha + \beta + \gamma = \frac{-q}{2}$; quo valore in secundâ substituto, re-
 sultat $\frac{q^2}{4} + 4 \times \alpha\beta + \alpha\gamma + \beta\gamma - \frac{q^2}{2} + s = 0$, unde $4 \times \alpha\beta + \alpha\gamma + \beta\gamma =$
 $\frac{q^2}{4} - s$; e tertiâ vero $\alpha\beta\gamma = \frac{r^3}{64}$; & consequenter æquatio, cujus ra-
 dices

dices sint α, β, γ , erit $z^3 + \frac{q}{2}z^2 + \frac{q^2}{16}z - \frac{s}{4}z - \frac{r^2}{64} = 0$, quatuor radices datæ æquationis erunt respective $\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma}; -\sqrt[3]{\alpha} - \sqrt[3]{\beta} + \sqrt[3]{\gamma}; -\sqrt[3]{\alpha} + \sqrt[3]{\beta} - \sqrt[3]{\gamma}; +\sqrt[3]{\alpha} - \sqrt[3]{\beta} - \sqrt[3]{\gamma}$; & æquatio, cujus radices sint prædictæ quantitates erit $z^3 - 2 \times \alpha + \beta + \gamma z^2 - 8 \sqrt[3]{\alpha\beta\gamma} z + \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\alpha\gamma - 2\beta\gamma = 0$.

Datâ vero unâ resolutione facile inveniri possint infinitæ aliæ, quæ rêvera eadem sunt ac data, forsan vero quâdem specie diversa esse videntur: Sint enim π, ρ, σ, τ , &c. tales functiones incognitarum datæ æquationis quantitatum, ut e quantitibus π, ρ, σ, τ , &c. per notas regulas inveniri possint illæ incognitæ quantitates; quibus inventis, & pro incognitis quantitibus datæ resolutionis substitutis, resultat resolutio quæsitâ. ex : g.

Datâ resolutione $x = \sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma}$, ubi α, β, γ sint radices datæ æquationis $z^3 + \frac{q}{2}z^2 + \frac{q^2}{16}z - \frac{s}{4}z - \frac{r^2}{64} = 0$; assumatur resolutio $v = \sqrt[2n]{\pi} + \sqrt[2n]{\rho} + \sqrt[2n]{\sigma}$; & erit $\pi = \alpha^n, \rho = \beta^n, \sigma = \gamma^n$; inveniatur igitur æquatio, cujus radices sint (n) potestates radicum æquationis $z^3 + \frac{q}{2}z^2 + \frac{q^2}{16}z - \frac{s}{4}z - \frac{r^2}{64} = 0$, & resultat æquatio, cujus radices sint π, ρ, σ ; si vero n fit 2, tum æquatio resultans erit $v^3 - \frac{s}{2} + \frac{q^2}{8}v^2 + \frac{qr^2}{64} + \frac{s^2}{16} - \frac{q^2s}{32} + \frac{q^4}{256} \times v - \frac{r^4}{4096} = 0$, & $x = \sqrt[4]{\pi} + \sqrt[4]{\rho} + \sqrt[4]{\sigma}$ erit radix æquationis $x^4 + qx^2 - rx + s = 0$; quatuor vero radices æquationis $x^4 + qx^2 - rx + s = 0$ erunt $\sqrt[4]{\pi} + \sqrt[4]{\rho} + \sqrt[4]{\sigma}, -\sqrt[4]{\pi} - \sqrt[4]{\rho} + \sqrt[4]{\sigma}, -\sqrt[4]{\pi} + \sqrt[4]{\rho} - \sqrt[4]{\sigma}, \sqrt[4]{\pi} - \sqrt[4]{\rho} - \sqrt[4]{\sigma}$.

Ex;

Ex. 3. Sit data æquatio $x^5 + qx^3 - rx^2 + sx - t = 0$, assumatur pro ejus resolutione $x = \sqrt[5]{\alpha} + \sqrt[5]{\beta} + \sqrt[5]{\gamma} + \sqrt[5]{\delta}$; ex calculo inveni æquationem biquadraticam, cujus radices sint generaliter $\alpha, \beta, \gamma, \delta$, irrationales quantitates necessario habere; & consequenter per hanc methodum haud resolvi potest data æquatio.

Hoc vero exemplum demonstrationem accipiat e subsequentibus principiis.

In datâ æquatione pro x substituendo ejus valorem assumptum $\sqrt[5]{\alpha} + \sqrt[5]{\beta} + \sqrt[5]{\gamma} + \sqrt[5]{\delta}$; e quantitatibus resultantibus inter se comparandis & valores coefficientum q, r, s, t e subsequentibus principiis acquisitos substituendo constabit exemplum.

Resolutio $\sqrt[5]{\alpha} + \sqrt[5]{\beta} + \sqrt[5]{\gamma} + \sqrt[5]{\delta}$ est resolutio æquationis $5 \times 5 \times 5 = 625$ dimensiones habentis, tot enim diversos valores habere possit prædicta resolutio: forsitan autem talis sit æquatio 625 dimensionum, ut admittat datam æquationem divisorem, i. e. (n) diversi valores resolutionis $\sqrt[5]{\alpha} + \sqrt[5]{\beta} + \sqrt[5]{\gamma} + \sqrt[5]{\delta}$ sint (n) radices datæ æquationis.

Ex. 4. Sit resolutio $x = a + b + c + d + e + f$; ita transformetur hæc æquatio ut exterminentur irrationales quantitates, & re-

sultat æquatio $x - f - a - b - c - d - e = 0$. Et sic de infinitis hujusmodi exemplis.

Cor. Datâ in genere (n) dimensionum æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, ejus resolutio erit resolutio (ut constat e præcedentibus) æquationis, cujus dimensiones sunt multo majores quam dimensiones datæ æquationis.

Hæc methodus resolutionem æquationum præbebit.

Assumatur

Assumatur resolutio, plures, quam datæ æquationis dimensiones (n), incognitas quantitates habens; transformetur hæc æquatio, ita ut exterminentur irrationales quantitates, & resultat æquatio, cujus data æquatio sit divisor; unde erui possunt (n) æquationes plures quam (n) incognitas quantitates habentes, quibus reductis resultat æquatio duas vel plures incognitas quantitates habens.

Incognitarum resultantis æquationis quantitatibus inveniuntur rationales radices, quibus pro suis valoribus substitutis, resultant incognitæ quantitates quæsitæ.

E præcedentibus exemplis, &c. conjicere liceat justam resolutionem, quam exigat quæcunque data æquatio; sed in æquationibus quinque & superiorum dimensionum calculus requiritur fere infinite laboriosus.

Hoc in loco videtur demonstrationem e diversis principiis petitam subungere, quod $\sqrt[5]{\alpha} + \sqrt[5]{\beta} + \sqrt[5]{\gamma} + \sqrt[5]{\delta}$ (ubi $\alpha, \beta, \gamma, \delta$ sint radices rationalis æquationis $x^4 + Px^3 + Qx^2 + Rx + S = 0$) haud sit generalis resolutio æquationis $x^5 + qx^3 - rx^2 + sx - t = 0$.

Sint $1, \pi, \rho, \sigma, \tau$ quinque radices æquationis $x^5 - 1 = 0$; & sit $\pi \times \rho = 1$, & $\sigma \times \tau = 1$: & 1^{mo} evanescant quantitates γ & δ ; & quinque radices datæ æquationis erunt per Cor. 2. Ex. 3. respective $\sqrt[5]{\alpha} + \sqrt[5]{\beta}$, $\pi \sqrt[5]{\alpha} + \rho \sqrt[5]{\beta}$, $\rho \sqrt[5]{\alpha} + \pi \sqrt[5]{\beta}$, $\sigma \sqrt[5]{\alpha} + \tau \sqrt[5]{\beta}$, $\tau \sqrt[5]{\alpha} + \sigma \sqrt[5]{\beta}$; & eodem modo cum evanescant quantitates β & δ inveniuntur radices $\sqrt[5]{\alpha} + \sqrt[5]{\gamma}$, $\pi \sqrt[5]{\alpha} + \rho \sqrt[5]{\gamma}$, $\rho \sqrt[5]{\alpha} + \pi \sqrt[5]{\gamma}$, $\sigma \sqrt[5]{\alpha} + \tau \sqrt[5]{\gamma}$, $\tau \sqrt[5]{\alpha} + \sigma \sqrt[5]{\gamma}$: cum vero evanescant quantitates α & δ inveniuntur radices $\sqrt[5]{\beta} + \sqrt[5]{\gamma}$, $\pi \sqrt[5]{\beta} + \rho \sqrt[5]{\gamma}$, $\rho \sqrt[5]{\beta} + \pi \sqrt[5]{\gamma}$, $\sigma \sqrt[5]{\beta} + \tau \sqrt[5]{\gamma}$, $\tau \sqrt[5]{\beta} + \sigma \sqrt[5]{\gamma}$.

Eadem autem inveniuntur coefficientes quantitatibus $\sqrt[5]{\alpha}$ & $\sqrt[5]{\beta}$, utrum evanescant quantitates $\sqrt[5]{\gamma}$ & $\sqrt[5]{\delta}$ necne; ergo, cum π sit co-

Q

efficiens

efficiens quantitatis $\sqrt[5]{a}$, ρ erit coëfficiens binarum quantitatum $\sqrt[5]{\beta}$ & $\sqrt[5]{\gamma}$; sed, cum ρ sit coëfficiens quantitatis $\sqrt[5]{\beta}$, π erit coëfficiens quantitatis $\sqrt[5]{\gamma}$; ergo coëfficiens quantitatis $\sqrt[5]{\gamma}$ debet esse simul π & ρ , quod est absurdum; & consequenter ex hac resolutione minime generaliter inveniri possunt radices æquationis $x^5 + qx^3 - rx^2 + sx - t = 0$.

Hoc vero aliter e subsequentibus principiis demonstrari possit.

Summa quinque radicum datæ æquationis, i. e. summa quinque valorum resolutionis $\sqrt[5]{a} + \sqrt[5]{\beta} + \sqrt[5]{\gamma} + \sqrt[5]{\delta}$ nihilo generaliter est æqualis; & consequenter summa quinque valorum ex singulâ quantitate $\sqrt[5]{a}$, $\sqrt[5]{\beta}$, $\sqrt[5]{\gamma}$, $\sqrt[5]{\delta}$ in eâ contentâ nihilo erit æqualis; quinque igitur valores quantitatis $\sqrt[5]{a}$ erunt respective $\sqrt[5]{a}$, $\pi \sqrt[5]{a}$, $\rho \sqrt[5]{a}$, $\sigma \sqrt[5]{a}$, $\tau \sqrt[5]{a}$; & sic de reliquis.

Deinde assumantur quicunque quinque valores hujusce generis resolutionis $\sqrt[5]{a} + \sqrt[5]{\beta} + \sqrt[5]{\gamma} + \sqrt[5]{\delta}$.

Ducantur quinque duo, tres & quatuor & quinque valores in sese; & aggregatum e singulis resultantibus contentis (qui fuerunt coëfficiētes q, r, s, t) literas a, β, γ, δ similiter involutas & haud evanescētes habere debet: sed facile constabit quod ex hac resolutione tales valores invenire impossibile sit.

Facile constat, ut hæc, quæ jam dicta fuerunt, applicari possunt in resolutionem $\sqrt[4]{a} + \sqrt[4]{\beta} + \sqrt[4]{\gamma}$ biquadraticarum æquationum; cum quatuor diversæ radices æquationis $x^4 - 1 = 0$ ducantur in quantitates $\sqrt[4]{a}$, $\sqrt[4]{\beta}$, $\sqrt[4]{\gamma}$.

SCHOLIUM.

Resolutio, quæ generaliter resolvēt quamcunque æquationem (n) dimensionum, necessario erit resolutio æquationis $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times \dots \times n$ dimensionum:

dimensionum: Hæc autem resolutio etiam inveniet, cum duæ radices, &c. n dimensionum fiant inter se æquales, ergo in generali resolutione æquationum superiorum dimensionum calculus requiritur pene infinite operosus. Frustra esset per tales methodos radices superioris æquationis $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$ quærere. Hinc recentiores, Vieta primum, deinde multi alii methodos approximationum invenere: valorem quam proxime radici æqualem primo cognitum pro concessio habuere, deinde valorem continuo ad radicem magis appropinquantem methodis fere iisdem investigavere omnes: minime autem pendet e ratione, quam habet quantitas assumpta pro radice ad radicem ipsam, seriei convergentia; potius vero pendet ex hoc, quod quantitas assumpta multo propior sit ad unam radicem quæsitam, quam ad ullam aliam datæ æquationis radicem, ex. grat.

Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. \dots Px^2 - Qx + R = 0$ cujus radices sint $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ sit etiam a quantitas assumpta pro radice α ; & transformetur data æquatio in alteram, cujus radices sint minores vel majores quam radices datæ æquationis per quantitatem (a), i. e. sint $\alpha - a, \beta - a, \gamma - a, \delta - a, \epsilon - a, \&c.$ & resultans æquatio erit

$$\begin{aligned} v^n + na v^{n-1} + n \times \frac{n-1}{2} a^2 v^{n-2} + \dots + na^{n-1} v + a^n \\ - p \quad - \frac{n-1}{1} ap \quad \dots \quad - \frac{n-1}{1} pa^{n-1} - pa^{n-1} \\ + \quad + q \quad \dots \quad + \frac{n-2}{1} qa^{n-1} + qa^{n-2} \\ - \frac{n-3}{1} ra^{n-1} - ra^{n-2} \\ \&c. \quad \&c. \end{aligned}$$

Sint $a : \alpha :: 1 : 1 + b$, $a : \beta :: 1 : 1 + k$, $a : \gamma :: 1 : 1 + l$, $a : \delta :: 1 : 1 + m$, $a : \epsilon :: 1 : 1 + n, \&c.$ & si modo a multo propior sit ad radicem α , quam ad β , vel γ , vel δ , vel $\epsilon, \&c.$ tum b multo minor erit quam k , vel l , vel m , vel $n, \&c.$ & radices resultantis æquationis sunt respectivæ $ba, ka, la, ma, na, \&c.$ ergo

$$ba \times ka \times la \times ma \times na \times \&c. = a^n - pa^{n-1} + qa^{n-2} - ra^{n-3} + sa^{n-4}$$

Q 2

$+sa^{n-4}\&c. = 2$ contentum sub singulis resultantis æquationis radicibus.
 $\& ka \times la \times ma \times na \times \&c. + ba \times ka \times la \times ma \times \&c. + ba \times la \times ma \times na \times \&c. +$
 $\&c. = na^{n-1} - \overline{n-1} pa^{n-2} + \overline{n-2} qa^{n-3} - \overline{n-3} ra^{n-4} + \&c. = P$
 aggregatum e singulis contentis sub $\overline{n-1}$ radicibus resultantis æqua-
 tionis: ergo approximatio per vulgares regulas inventa
 $\frac{a^n - pa^{n-1} + qa^{n-2} - ra^{n-3} + \&c.}{na^{n-1} - \overline{n-1} pa^{n-2} + \overline{n-2} qa^{n-3} - \&c.} : \text{ad quantitatem quæsitam } \alpha - a$
 $= b :: klmn \&c. \left(\frac{2}{b} \right) : klmn \&c. + bklm \&c. + blmn \&c. +$
 $\&c. (P).$

Si vero quantitas assumpta multo propior sit ad duas radices α & β
 datæ æquationis quam ad reliquas, i. e. duæ radices resultantis æqua-
 tionis, quæ sit $v^n \dots Tv^4 - Sv^3 + Rv^2 - Qv + P = 0$, sint mi-
 nores quam reliquæ; tum radices quadraticæ æquationis $v^2 - \frac{2}{R}v +$
 $\frac{P}{R} = 0$ erunt quam proxime illæ radices quæsitæ; si vero tres, tum ra-
 dices æquationis $v^3 - \frac{R}{S}v^2 + \frac{Q}{S}v - \frac{P}{S} = 0$ erunt quam proxime ra-
 dices quæsitæ; & sic deinceps: & facile confimili methodo inveniri
 possint rationes, quas habent radices inventæ per has regulas ad veras
 radices quæsitæ.

Sed plura de infinitis seriebus, quam ratio nostri instituti exigit, ad-
 jiciuntur: haud enim generalem tractatum recipiant, ni de fluxionibus
 &c. agatur.

C A P. IV.

DE duabus vel pluribus æquationibus duas vel plures incognitas quantitates habentibus.

P R O B. XXVII.

Duas æquationes in unam transformare, ita ut incognita quantitas exterminetur.

Collocentur singulæ æquationis termini juxta incognitæ quantitatis exterminandæ dimensiones, ita subducatur alterutra æquatio aut ejus multiplex de alterâ, & reliquum aut ejus multiplex de ablatâ æquatione, & sic deinceps, ut tandem exterminetur incognita quantitas; i. e. exterminetur incognita quantitas eodem modo, quo inveniuntur communes divisores.

Ex. 1. Sint duæ æquationes $y^2 + 3 + 2x \times y + 4x^2 + 5x + 6 = 0$, & $y^2 + 5 + 2xy + 10x^2 + 7x + 12 = 0$, & operatio erit hujusmodi

$$y^2 + 3 + 2xy + 4x^2 + 5x + 6) y^2 + 5 + 2xy + 10x^2 + 7x + 12) 1$$

$$y^2 + 3 + 2xy + 4x^2 + 5x + 6$$

$$2y + 6x^2 + 2x + 6 = 0,$$

$$2y + 6x^2 + 2x + 6) y^2 + 3 + 2xy + 4x^2 + 5x + 6) \frac{1}{2} y + \frac{x - 3x^2}{2}$$

$$y^2 + 3x^2 + x + 3y$$

$$x - 3x^2 \times y + 4x^2 + 5x + 6$$

$$x - 3x^2 \times y + \frac{x - 3x^2}{2} \times 6x^2 + 2x + 6$$

$$4x^2 + 5x + 6 - x - 3x^2 \times 3x^2 + x + 3$$

et ita transformantur duæ æquationes in unam ($4x^2 + 5x + 6 - x - 3x^2 \times 3x^2 + x + 3 = 0$), ut exterminetur incognita quantitas (y).

Ex. 2. Sint duæ æquationes $ay^2 + by + c = 0$, & $py^2 + qy + r = 0$; eas in unam reducere, ita ut incognita quantitas (y) exterminetur.

Ducatur

Ducatur prior æquatio in p , posterior vero in a coefficientes terminorum, in quibus maximæ inveniuntur dimensiones quantitatis (y) exterminandæ, causâ fractiones evitandi; & operatio erit

$$pay^3 + pby + pc = 0$$

$$pay^3 + qay + ra = 0 \text{ subducatur posterior de priori, \&}$$

residuum erit $pb - qay + pc - ra = 0$: ducantur æquationes $py^3 + qy + r = 0$, & $pb - qay + pc - ra = 0$ causâ evitandi fractiones in $pb - qa$ & p respective, & resultant

$$p^2b - pqay^2 + qp^2b - q^2ay + pbr - qar = 0, \&$$

$p^2b - pqay^2 + p^2c - pra \times y$ subducatur posterior de priori, & erit residuum $qp^2b - q^2a + pra - p^2cy + pbr - qar$, & eodem processu repetito exterminabitur quantitas (y), & resultat quantitas quæsitæ

$$pc - ra \times qp^2b - q^2a + pra - p^2c - pb - qa \times pbr - qar = 0. \\ \text{Eodem modo sint duæ æquationes } ay^3 + by + c = 0, \& py^3 + qy^2 + ry + s = 0, \& \text{ exterminetur incognita quantitas } y, \& \text{ resultabit} \\ \text{æquatio } aB - bA \times cB - asb + sa^2 - cA = 0, \text{ ubi } A = aq - pb, B = ar - pc.$$

Sint etiam $ax^3 + bx^2 + cx + d = 0$, & $px^3 + qx^2 + rx + s = 0$; & supponantur $A = bp - aq$, $B = dp - as$, $C = cp - ar$, $P = cA - aAB - bAC + aC^2$, $Q = cAB - aBB - dAC$, $R = dA - bAB + aBC$; tum $PQ + RR = 0$.

Cor. Æquationes sic derivatæ plerumque plures habent dimensiones quam necessario exigit problema.

Sint enim duæ æquationes $Ay^m + By^{m-1} + Cy^{m-2} + Dy^{m-3} + \&c. = 0$, $ay^m + by^{m-1} + cy^{m-2} + dy^{m-3} + \&c. = 0$, ubi $A, B, C, D, \&c.$ $a, b, c, d, \&c.$ sint quæcunque functiones incognitæ quantitatis (x); ducatur prior æquatio in $x - px^{m-1} + qx^{m-2} - \&c.$ & duæ æquationes $x^m - px^{m-1} + qx^{m-2} - \&c. \times Ay^m + By^{m-1} + Cy^{m-2} + \&c. = 0$, & $ay^m + by^{m-1} + cy^{m-2} + dy^{m-3} + \&c. = 0$ habent (r) radices incognitæ quantitatis (x), $m \times r$ radices vero incognitæ quantitatis (y), quæ
haud

haud inveniuntur in duabus datis æquationibus; omnis enim radix æquationis $x^r - px^{r-1} + qx^{r-2} - \&c. = 0$ pro (x) substituta in æquatione $x^r - px^{r-1} + qx^{r-2} - \&c. \times Ay^m + By^{m-1} + Cy^{m-2} + Dy^{m-3} + \&c. = 0$ æquationem nihilo æqualem reddet; si vero in posteriori æquatione $ay^m + by^{m-1} + cy^{m-2} + dy^{m-3} + \&c. = 0$ pro incognitâ quantitate (x) substituaturs prædicta radix, resultat æquatio, quæ (m) habet dimensiones incognitæ quantitatibus (y) , ergo m sunt valores incognitæ quantitatibus (y) , quæ correspondent unicuique e prædictis radicibus æquationis $x^r - px^{r-1} + qx^{r-2} - \&c. = 0$, & consequenter $m \times r$ inveniuntur valores incognitæ quantitatibus (y) in his duabus æquationibus, quæ in datis haud inveniuntur.

Si vero $x^r - px^{r-1} + qx^{r-2} - \&c.$ & a communem habeant divisorem, qui sit $x^s - Px^{s-1} + Qx^{s-2} - \&c.$ tum haud plures novæ radices incognitæ quantitatibus (y) inveniuntur, quam $r - s \times m + s \times m - 1$; si vero tres quantitates $x^r - px^{r-1} + qx^{r-2} - \&c.$ & a & b habeant communem divisorem, qui sit $x^t - \pi x^{t-1} + \rho x^{t-2} - \&c.$ tum haud plures inveniuntur novæ radices incognitæ quantitatibus (y) quam $r - s \times m + s - t \times m - 1 + t \times m - 2$; & sic deinceps; novæ autem radices incognitæ quantitatibus (x) in omnibus his casibus eadem erunt, erunt enim radices æquationis $x^r - px^{r-1} + qx^{r-2} - \&c. = 0$, ni quædam radices æquationis $x^r - px^{r-1} + qx^{r-2} - \&c. = 0$ (ut facile e prædictâ methodo constabit) nullos habeant correspondentes valores incognitæ quantitatibus (y) .

Ex. Sint duæ datæ æquationes $y^n + a + bx y^{n-1} + c + dx + ex^2 y^{n-2} + \&c. = 0$, & $A + Bx y^{n-1} + C + Dx + Ex^2 y^{n-2} + \&c.$ ducatur prior æquatio in $A + Bx$, posterior vero in y , & resultant æquationes $A + Bx y^n + A + Bx \times a + bx y^{n-1} + A + Bx \times c + dx + ex^2 y^{n-2} + \&c. = 0$, & $A + Bx y^n + C + Dx + Ex^2 y^{n-1} + \&c. = 0$, quarum differentia erit $A + Bx \times a + bx - C - Dx - Ex^2 y^{n-1} + \&c. = 0$, duæ æquationes $A + Bx \times a + bx - C - Dx - Ex^2 y^{n-1} + \&c. = 0$, & $A + Bx y^{n-1} + C + Dx + Ex^2 y^{n-2} + \&c. = 0$ habent omnes valores

lores incognitæ quantitatis (x), quos habent duæ datæ æquationes & habent etiam unum valorem $A + Bx = 0$, i. e. $x = \frac{-B}{A}$, quem haud habent datæ æquationes: habent etiam prædictæ æquationes ($n - 2$) valores incognitæ quantitatis (y), quos non habent datæ æquationes, ni $A + Bx$ fit divisor quantitatis $C + Dx + Ex^2$.

Hinc constat, ut ex hac methodo reductionis novæ radices in resultantibus æquationibus semper orientur, quæ in datis æquationibus haud continentur; & facile etiam e reductione constabit æquatio, cujus radices sint novæ radices prædictæ.

Cor. 2. Sint duæ æquationes $x^r + ax^{r-1} + bx^{r-2} + \&c. \times y^n + dx^{r+1} + ex^r + fx^{r-1} + \&c. y^{n-1} + \&c. = 0$, & $x^r + ax^{r-1} + bx^{r-2} + \&c. y^n + dx^{r+1} + ex^r + fx^{r-1} + \&c. \times y^{n-1} + \&c. = 0$; & per methodum in problemate contentam ita reducantur æquationes ut resultant duæ æquationes, quæ haud majores habent dimensiones incognitæ quantitatis (y), quam $n - 1$; & dimensiones quantitatis (x), quæ ducantur in (y^{n-1}), haud majores erunt in unâ quam $r + s + 1$, in alterâ quam $2r + s + 2$; reducantur duæ datæ æquationes, in duas alias, in quibus dimensiones incognitæ quantitatis (y) sint ($n - 2$), & dimensiones incognitæ quantitatis (x) quæ ducantur in (y^{n-1}) haud majores erunt in unâ quam $r + s + 1 + 2r + s + 2 + 1 = 3r + 2s + 4$, in alterâ quam $3r + 2s + 4 + 2r + s + 2 + 1 = 4r + 3s + 6$; proximæ autem dimensiones incognitæ quantitatis (x) haud majores erunt quam $3r + 2s + 4 + 4r + 3s + 6 + 1 = 7r + 5s + 11$ in unâ, $7r + 5s + 11 + 3r + 2s + 4 + 1 = 10r + 7s + 15$ in alterâ; & sic deinceps.

Cor. 3. Sint duæ datæ æquationes $y^n + ay^{n-1} + by^{n-2} + cy^{n-3} + dy^{n-4} + ey^{n-5} + fy^{n-6} + \&c. = 0$, & $y^m + Ay^{m-1} + By^{m-2} + Cy^{m-3} + Dy^{m-4} + Ey^{m-5} + Fy^{m-6} + \&c. = 0$, quarum dimensiones (n) majores sint quam m ; & per prædictam methodum constat, ut reduci possint in duas æquationes (m & n) dimensionum $y^m + Ay^{m-1} + By^{m-2} + Cy^{m-3}$

$+ Cy^{m-1} + \&c. = 0$, & $P \times y^{n-1} + Q \times y^{n-2} + \&c. = 0$; ubi s haud major sit quam $n - m + 1$; si resultantes æquationes easdem & nullas alias habent radices præter eas, quas habent datæ æquationes.

Sit $s = n - m + 1$, & facile constat æquatio resultans; omnes enim numerales coefficientes erunt 1; coefficientis autem quantitatis y^{n-1} oritur e singulis contentis coefficientium datarum æquationum, quæ possunt conficere s dimensiones: si modo contentum literale $a^\alpha b^\beta c^\gamma d^\delta \&c. \times A^\pi B^\rho C^\sigma D^\tau \&c.$ habeat $\alpha + 2\beta + 3\gamma + 4\delta + \&c. + \pi + 2\rho + 3\sigma + 4\tau + \&c.$ dimensiones.

Coefficientis quantitatis y^{n-1} oritur e singulis contentis coefficientium prædictarum, quæ possunt conficere $s+1$ dimensiones, & sic deinceps: & sic de earum signis.

Cor. 4. Hinc inveniri potest, utrum duæ vel plures æquationes aliquid absurdi in se contineat.

Dividantur æquationes, ita ut tandem exterminentur incognitæ quantitates; & si modo inæquales quantitates tandem æquales ex hypothesi emergant, aliquid absurdi in se continent datæ æquationes; sin aliter vero non. e. g. Sint æquationes $y^2 + 7x^2 + 20 = 0$, & $y^2 + 7x^2 + 40 = 0$; subtrahatur prior æquatio e posteriori, & resultat $40 - 20 = 0$, i. e. $40 = 20$, q. e. absurdum.

Omnia hæc etiam applicari possunt in (n) æquationes (m) incognitas quantitates habentes.

Cor. 5. Constat, ut hæc methodus reducendi duas vel plures æquationes duas vel plures dimensiones habentes in unam, ita ut exterminentur incognitæ quantitates, eadem est ac methodus reducendi duas vel plures simplices æquationes unam solummodo dimensionem incognitarum quantitatum habentes in unam, ita ut exterminentur incognitæ quantitates. Sint enim duæ æquationes $Ay^m + By^{m-1} + Cy^{m-2} + \&c. = 0$, & $ay^n + by^{n-1} + cy^{n-2} + dy^{n-3} + \&c. = 0$; eas in unam reducere, ita ut exterminetur incognita quantitas (y) .

Supponantur etiam quantitates $y^m, y^{m-1}, y^{m-2}, \&c. y^n, y^{n-1}, y^{n-2}, \&c.$

R

tanquam

tanquam diversæ incognitæ quantitates; & reducantur eodem modo, quo simplices æquationes diversas incognitas quantitates habentes reducantur; & fere eadem erit methodus reducendi has æquationes ac præcedens.

2. Transformare duas æquationes ex alia methodo in unam, ut incognita quantitas exterminetur.

Sint duæ æquationes

$$y^n - py^{n-1} + qy^{n-2} - ry^{n-3} + sy^{n-4} - \&c. = 0,$$

$$\& V - Ty + Sy^2 - Ry^3 + Qy^4 - \&c. = 0.$$

Sint etiam $\alpha, \beta, \gamma, \delta, \&c.$ prioris æquationis radices (y), quarum numerus erit (n), quibus in posteriori pro suo valore substitutis, resultant quantitates

$$V - T\alpha + S\alpha^2 - R\alpha^3 + Q\alpha^4 - \&c.$$

$$V - T\beta + S\beta^2 - R\beta^3 + Q\beta^4 - \&c.$$

$$V - T\gamma + S\gamma^2 - R\gamma^3 + Q\gamma^4 - \&c.$$

$$V - T\delta + S\delta^2 - R\delta^3 + Q\delta^4 - \&c.$$

&c.

Multiplicentur hæ quantitates resultantes continuo in sese, & contentum erit

$$V^n - V^{n-1}T \times \alpha + V^{n-2}S \times \alpha^2 + V^{n-3}R \times \alpha^3 + \&c.$$

β

β^2

β^3

γ

γ^2

γ^3

δ

δ^2

δ^3

&c.

&c.

&c.

$$+ V^{n-2}T^2 \times \alpha\beta + V^{n-3}TS \times \alpha^2\beta + \&c.$$

$\alpha\gamma$

$\beta^2\alpha$

$\beta\gamma$

$\alpha^2\gamma$

$\alpha\delta$

$\gamma^2\alpha$

$\beta\delta$

$\beta^2\gamma$

$\gamma\delta$

$\gamma^2\beta$

&c.

&c.

Ope problematum primi & tertii inveniri potest hoc contentum, quod erit

$V^n -$

$$V^n - V^{n-1}Tp + V^{n-2}S \times p^2 - 2q + V^{n-3}R \times p^3 - 3pq + 3r + \&c. \\ + V^{n-2}T^2 \times q + V^{n-3}TS \times pq - 3r + \&c.$$

nihilo fiat æquale, & transformantur duæ æquationes in unam, ut exterminetur incognita quantitas (y).

Ex. 1. Sint duæ æquationes $y^2 + 3 + 2x \times y + 4x^2 + 5x + 6 = 0$,
& $y^2 + 5 + 2x \times y + 10x^2 + 7x + 12 = 0$.

Sint α & β prioris æquationis radices (y), & iisdem pro (y) in posteriori æquatione substitutis, resultant quantitates

$$\alpha^2 + 5 + 2x \times \alpha + 10x^2 + 7x + 12$$

& $\beta^2 + 5 + 2x \times \beta + 10x^2 + 7x + 12$ quibus quantitatibus in sese ductis, fit rectangulum

$$\alpha^2\beta^2 + 5 + 2x \times \alpha\beta \times \alpha + \beta + 10x^2 + 7x + 12 \times \alpha^2 + \beta^2 + 5 + 2x \times \alpha\beta + 10x^2 + 7x + 12 \times 5 + 2x \times \alpha + \beta + 10x^2 + 7x + 12$$

ope problematis primi prodit hoc rectangulum

$$4x^2 + 5x + 6 - 5 + 2x \times 4x^2 + 5x + 6 \times 3 + 2x + 10x^2 + 7x + 12 \times 3 + 2x - 2 \times 6x^2 + 5x + 6 + \\ 5 + 2x \times 4x^2 + 5x + 6 - 10x^2 + 7x + 12 \times 5 + 2x \times 3 + 2x \times 10x^2 + 7x + 12.$$

Fiat hoc rectangulum nihilo æquale, & erit æquatio quæsitæ. ^b

Cor. 1. Facile constat, quod tot erunt valores incognitarum quantitatium, quot invenit hæc reductionis methodus, plures autem minime admittant datæ æquationes.

Cor. 2. Hac methodo transformentur duæ æquationes, quarum dimensiones sunt n & m respectivæ, duas incognitas quantitates (x & y) habentes in unam, ut exterminetur una incognita quantitas (y), & plures quam ($n \times m$) dimensiones haud habere potest resultans æquatio.

Cor. 3. Sint duæ æquationes

$$My^n + a + bx \times y^{n-1} + c + dx + ex^2 \times y^{n-2} + f + gx + hx^2 + kx^3 \times y^{n-3} + \&c. = 0, \\ Ny^m + A + Bx \times y^{m-1} + C + Dx + Ex^2 \times y^{m-2} + F + Gx + Hx^2 + Kx^3 \times y^{m-3} + \&c. = 0,$$

quarum dimensiones sunt n & m , duas incognitas quantitates (x & y) habentes; & rejiciantur harum æquationum omnes termini, in quibus incognitarum quantitatium (x & y) dimensio haud invenitur max-

ima, & quantitates resultantes erunt

$$My^n + bx^{n-1}y^{n-1} + ex^2y^{n-2} + kx^3y^{n-3} + \&c.$$

$$Ny^m + Bxy^{m-1} + Ex^2y^{m-2} + Kx^3y^{m-3} + \&c.$$

si communem habeant divisorem hæ quantitates resultantes, $(n \times m)$ dimensiones non habere potest resultans æquatio, cujus radix vel sit incognita quantitas (x vel y); si vero nullum habeant communem divisorem, $(n \times m)$ dimensiones habebit æquatio resultans.

Sint tres vel plures æquationes, tot incognitas ($x, y, z, v, \&c.$) quantitates habentes, quarum dimensiones sint respective $n, m, o, l, \&c.$ & rejiciantur harum æquationum omnes termini, in quibus incognitarum quantitarum ($x, y, z, v, \&c.$) dimensio haud invenitur maxima: & si quantitarum resultantium nullus invenitur communis divisor, vel ex iis additione, subtractione, &c. formari possit divisor, qui duabus diversis resultantibus quantitatibus communis est; tum æquatio, cujus radix sit ($x, y, z, v, \&c.$) $n \times m \times o \times l \times \&c.$ dimensiones habet; sin aliter vero non.

Cor. 4. Si æquatio resultans, cujus radix est incognita quantitas (x vel y), dimensiones $(n \times m)$ habeat: summa radicum solummodo pendet e terminis duarum datarum æquationum, in quibus incognitarum quantitarum (x & y) dimensiones inveniuntur n & $n-1$, m & $m-1$ respective, i. e. pendet e terminis in priori æquatione

$$My^n + bx^{n-1}y^{n-1} + ex^2y^{n-2} + kx^3y^{n-3} + \&c.$$

$$\& ay^{n-1} + dx^{n-2}y^{n-2} + bx^2y^{n-3} + \&c.$$

& terminis in posteriori æquatione

$$Ny^m + Bxy^{m-1} + Ex^2y^{m-2} + Kx^3y^{m-3} + \&c.$$

$$Ay^{m-1} + Dx^{m-2}y^{m-2} + Hx^2y^{m-3} + \&c.$$

Summa rectorum sub singulis binis pendet e terminis $n, n-1$ & $n-2$ dimensionum in unâ æquatione; & terminis $m, m-1$ & $m-2$ dimensionum in alterâ; i. e. terminis in priori æquatione

$$My^n + bx^{n-1}y^{n-1} + ex^2y^{n-2} + kx^3y^{n-3} + \&c.$$

$$ay^{n-1} + dx^{n-2}y^{n-2} + bx^2y^{n-3} + \&c.$$

$$\& cy^{n-2} + gx^{n-3}y^{n-3} + \&c.$$

terminis

terminis autem in posteriori æquatione

$$Ny^m + Bxy^{m-1} + Ex^2y^{m-2} + Kx^3y^{m-3} + \&c.$$

$$Ay^{m-1} + Dxy^{m-2} + Hx^2y^{m-3} + \&c.$$

$$\& Cy^{m-2} + Gxy^{m-3} + \&c.$$

& sic deinceps de summâ contentorum sub singulis ternis, &c.

Cor. 5. Sint duæ æquationes n & m dimensionum

$$ax^r + bx^{r-1} + cx^{r-2} + \&c. \times y^{n-r} + dx^{r+1} + ex^r + \&c. \times y^{n-r-1} + \&c. = 0.$$

$$\& Ax^s + Bx^{s-1} + \&c. \times y^{m-s} + Dx^{s+1} + Ex^s + \&c. \times y^{m-s-1} + \&c. = 0.$$

Ita reducantur duæ æquationes in unam, ut exterminetur incognita quantitas (y), & resultans æquatio haud plures habere possit dimensiones quam $(nm - sr)$: facile e prædictis constat, utrum tot vel minores habeat dimensiones.

Cor. 6. Sint tres vel plures (α) æquationes tres vel plures incognitas quantitates x, y, z, v , &c. habentes, & quarum dimensiones sint respective n, m, o, r , &c. per hanc methodum exterminentur incognitæ quantitates x, y, z , &c. e respectivis æquationibus successive; & æquatio resultans, cujus radix sit v , dimensiones $n^{\alpha-1} \times m^{\alpha-2} \times o^{\alpha-3} \times \&c.$ habere possit.

3. Sint duæ datæ æquationes duas incognitas quantitates x & y habentes, inveniantur e notis regulis (si tales dentur regulæ) ex unâ datâ æquatione omnes diversi valores unius incognitæ quantitatæ (x) terminis vero alterius.

Substituantur hi diversi valores pro incognitâ quantitate (x) in alterâ datâ æquatione, & ducantur quantitates exinde resultantes continuo in sese; & ita reducuntur duæ æquationes in unam ut exterminetur incognita quantitas (x).

Ex. 1. Sint duæ æquationes $x^2 - 2ax + a^2 - b = 0$, & $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$; duæ radices prioris æquationis erunt $x = a \pm \sqrt{b}$; substituantur hæ radices pro suô valore (x) in æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, & ducantur duæ quantitates resultantes in sese, & scribantur

$$A = a + b^{\frac{1}{2}} + a - b^{\frac{1}{2}} - p \times a + b^{\frac{1}{2}} + a - b^{\frac{1}{2}} + q \times a + b^{\frac{1}{2}} + a - b^{\frac{1}{2}} + \&c.$$

$$B =$$

$B = a + b^{\frac{1}{2}} - a - b^{\frac{1}{2}} - p \times a + b^{\frac{1}{2}} - a - b^{\frac{1}{2}} + q \times a + b^{\frac{1}{2}} - a - b^{\frac{1}{2}} + \&c.$
 & æquatio resultans erit $A^2 - B^2 = 0$.

Ex. 2. Sint duæ æquationes $x^3 - 3ax^2 + 3a^2 + bx - a^3 - ba + c = 0$;
 & $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + \&c. = 0$; tres au-
 tem radices prioris æquationis erunt

$$a + \sqrt[3]{-\frac{1}{2}c + \sqrt{\frac{1}{4}c^2 + \frac{b^3}{27}}} (a) + \sqrt[3]{-\frac{1}{2}c - \sqrt{\frac{1}{4}c^2 + \frac{b^3}{27}}} (\beta), a - \frac{1 + \sqrt{-3}}{2}$$

$a - \frac{1 - \sqrt{-3}}{2} \beta, a - \frac{1 - \sqrt{-3}}{2} \alpha - \frac{1 + \sqrt{-3}}{2} \beta$; substituantur
 hæ tres radices pro suo valore (x) in æquatione $x^n - px^{n-1} + qx^{n-2} -$
 $rx^{n-3} + \&c. = 0$, & ducantur tres quantitates resultantes continuo
 in sese; et supponantur

$$A = a + \alpha + \beta + a - \frac{1 + \sqrt{-3}}{2} \alpha - \frac{1 - \sqrt{-3}}{2} \beta + a - \frac{1 - \sqrt{-3}}{2} \alpha$$

$$- \frac{1 + \sqrt{-3}}{2} \beta - p \times a + \alpha + \beta + a - \frac{1 + \sqrt{-3}}{2} \alpha - \frac{1 - \sqrt{-3}}{2} \beta +$$

$$a - \frac{1 - \sqrt{-3}}{2} \alpha - \frac{1 + \sqrt{-3}}{2} \beta + q \times a + \alpha + \beta + \&c. \&c.$$

$$B = a + \alpha + \beta - \frac{1 + \sqrt{-3}}{2} \alpha - \frac{1 + \sqrt{-3}}{2} \alpha - \frac{1 - \sqrt{-3}}{2} \beta - \frac{1 - \sqrt{-3}}{2} \beta$$

$$\times a - \frac{1 - \sqrt{-3}}{2} \alpha - \frac{1 + \sqrt{-3}}{2} \beta - p \times a + \alpha + \beta - \frac{1 + \sqrt{-3}}{2} \alpha - \frac{1 - \sqrt{-3}}{2} \beta$$

$$+ q \&c.$$

$$C = a + \alpha + \beta - \frac{1 - \sqrt{-3}}{2} \alpha - \frac{1 + \sqrt{-3}}{2} \alpha - \frac{1 - \sqrt{-3}}{2} \beta - \frac{1 + \sqrt{-3}}{2} \beta$$

$$= B$$

$$\frac{a - \frac{1 - \sqrt{-3}}{2} \alpha - \frac{1 + \sqrt{-3}}{2} \beta}{\frac{1 - \sqrt{-3}}{2} \alpha - \frac{1 + \sqrt{-3}}{2} \beta} = \frac{a - \frac{1 - \sqrt{-3}}{2} \alpha - \frac{1 + \sqrt{-3}}{2} \beta}{\frac{1 - \sqrt{-3}}{2} \alpha - \frac{1 + \sqrt{-3}}{2} \beta} + q \&c.$$

& æquatio resultans erit

$$A^3 - 3ABC + B^3 + C^3 = a. \text{ Et sic deinceps.}$$

Cor. Cum vero tres vel plures dentur æquationes tres vel plures incognitas quantitates habentes, ex hac & præcedenti methodo eadem resultabunt æquationes.

4. Sint duæ æquationes $Ay^m + By^{m-1} + Cy^{m-2} + \&c. = 0$, & $ay^n + by^{n-1} + cy^{n-2} + \&c. = 0$; assumantur duæ quantitates $P y^{m-1} + Q y^{m-2} + R y^{m-3} + \&c.$ & $p y^{n-1} + q y^{n-2} + r y^{n-3} + \&c.$

Ducatur data æquatio $Ay^m + By^{m-1} + Cy^{m-2} + \&c. = 0$ in assumptam quantitatem $p y^{n-1} + q y^{n-2} + r y^{n-3} + \&c. = 0$, & supponatur rectanguli resultantis ($Apy^{m+n-1} + Aq + Bp \times y^{m+n-2} + Ar + Bq + Cp \times y^{m+n-3} + \&c. = 0$) singulus terminus æqualis ejus correspondenti termino in æquatione e multiplicatione datæ æquationis $ay^n + by^{n-1} + cy^{n-2} + \&c.$ in assumptam quantitatem $P y^{m-1} + Q y^{m-2} + R y^{m-3} + \&c.$ i.e. $aPy^{m+n-1} + aQ + bP y^{m+n-2} + aR + bQ + cP y^{m+n-3} + \&c.$ unde $Ap - aP = 0$ & $Aq + Bp - aQ - bP = 0$, &c. & ex simplicibus æquationibus resultantibus, quarum numerus est $m + n$, & quæ habent $m + n$ incognitas quantitates $P, Q, R, \&c. p, q, r, \&c.$ erui possint illæ incognitæ quantitates $P, Q, R, \&c. p, q, r, \&c.$

Hæc methodus etiam applicari possit in tres vel plures (n) æquationes tot incognitas quantitates habentes.

Facile enim constat ex his duabus observationibus numerus dimensionum, quas habere debet incognita quantitas in assumptis æquationibus; nempe quod singula æquatio in suam correspondentem æquationem assumptam ducta semper eandem conficiet summam, & quod dimensiones æquationum assumptarum simulque adjectarum æquales erunt differentia inter prædictam summam & numerum $(n-1)$ datarum æquationum unitate diminutum.

5. Est

5. Est etiam alia methodus transformandi duas æquationes in unam, ita ut incognita quantitas exterminetur.

Per methodum serierum infinitarum inveniuntur singulæ radices incognitæ quantitatis exterminandæ in unâ datâ æquatione, quibus pro suo valore in alterâ substitutis, & quantitatibus resultantibus in sese multiplicatis, contentum nihilo fiat æquale & erit quæsitæ æquatio.

Ex. Sint duæ æquationes $y^2 - 4x + 2xy + 3x^2 + 4x + 4 = 0$
 & $y^2 - 3xy + 4x^2 + 6 = 0$
 eas in unam transformare, ut exterminetur incognita quantitas (y).

Incognitæ quantitatis (y) duæ radices in priori æquatione per infinitas series inveniuntur, viz.

$$y = x + 1 + \frac{3}{2x} + \frac{9}{8x^2} + \&c.$$

$$y = 3x + 1 - \frac{3}{2x} - \frac{9}{8x^2} - \&c.$$

quibus radicibus pro (y) in posteriori æquatione substitutis, resultant quantitates.

$$2x^2 - x + \frac{11}{2} + \frac{3}{x} + \frac{9}{8x^2} + \&c.$$

$$\& 4x^2 + 3x + \frac{5}{2} - \frac{3}{x} - \frac{9}{8x^2} - \&c.$$

Ducantur hæ duæ quantitates resultantes in sese, & rectangulum sub iis contentum, erit

$$8x^4 + 2x^3 + 24x^2 + 20x + 28 = 0,$$

quod nihilo fiat æquale & erit æquatio quæsitæ

$$8x^4 + 2x^3 + 24x^2 + 20x + 28 = 0;$$

cujus radices omnes erunt impossibiles.

Omnia corollaria e priori methodo deducta ex hac etiam investigari possint.

6. Est etiam alia methodus hoc problema resolvendi.

Sint duæ æquationes

$$Ay^n + a + bx y^{n-1} + c + dx + ex^2 y^{n-2} + \&c. = 0,$$

&

& $Ny^m + \overline{A + Bx}y^{m-1} + \overline{C + Dx + Ex^2}y^{m-2} + \&c. = 0$,
 eas in unam transformare, ita ut exterminetur incognita quantitas (x).

Ducantur duæ datæ æquationes $ny^n + \overline{a + bx}y^{n-1} + \&c. = 0$,
 $Ny^m + \overline{A + Bx}y^{m-1} + \&c. = 0$ in duas assumptas $my^{m-n} +$
 $\overline{p + qx}y^{m-n-1} + \overline{r + sx + tx^2}y^{m-n-2} + \&c. = 0$, & $My^{m-n} +$
 $\overline{P + Qx}y^{m-n-1} + \overline{R + Sx + Tx^2}y^{m-n-2} + \&c. = 0$, respec-
 tive; fiant æquationes resultantes inter se æquales, & ita capiantur
 coefficientes assumptarum æquationum ut evanescant omnes ter-
 mini resultantis æquationis; iis exceptis, in quibus solummodo in-
 veniuntur incognita quantitas (y), & ejus Potestates.

Hoc perficitur simplicium æquationum solummodo ope.

Semper tot vel plures inveniantur incognitæ quantitates, quot
 resultant simplices æquationes.

PROB. XXVIII.

*Datâ æquatione unam vel plures incognitas quantitates habente, inve-
 nire constitutionem ejus coefficientium.*

1. Datâ æquatione ($x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$),
 unam incognitam quantitatem (x) habente, invenire ejus Coeffici-
 entium ($p, q, r, s, \&c.$) constitutionem.

Sint $\alpha, \beta, \gamma, \delta, \&c.$ Radices datæ æquationis, quarum numerus fit
 n , quibus pro suo valore (x) substitutis, resultant æquationes
 $\alpha^n - p\alpha^{n-1} + q\alpha^{n-2} - r\alpha^{n-3} + s\alpha^{n-4} - \&c. = 0$, & $\beta^n - p\beta^{n-1} +$
 $q\beta^{n-2} - r\beta^{n-3} + s\beta^{n-4} - \&c. = 0$, & $\gamma^n - p\gamma^{n-1} + q\gamma^{n-2} - r\gamma^{n-3} +$
 $s\gamma^{n-4} - \&c. = 0$, & $\delta^n - p\delta^{n-1} + q\delta^{n-2} - r\delta^{n-3} + s\delta^{n-4} - \&c. = 0$, &c.
 reducantur hæ æquationes in unam ita ut exterminentur omnes
 præter unam (p), cujus constitutio quæritur; & invenietur $p = \alpha +$
 $\beta + \gamma + \delta + \&c.$; & eodem modo invenietur $q = \alpha\beta + \alpha\gamma + \beta\gamma +$
 $\alpha\delta + \&c.$ $r = \alpha\beta\gamma + \alpha\beta\delta + \&c.$ & sic deinceps.

2. Datâ æquatione ($y^n + \overline{a + bx}y^{n-1} + \overline{c + dx + ex^2}y^{n-2} +$
 $\overline{f + gx + hx^2 + kx^3}y^{n-3} + \&c. = 0$ duas incognitas quantitates
 (x & y) habente, invenire ejus coefficientium constitutionem.

S

Sint

Sint incognitæ quantitatis (y) radices $\alpha, \beta, \gamma, \delta, \&c.$ & incognitæ quantitatis (x) correspondentes radices $\pi, \rho, \sigma, \tau, \&c.$ & singulæ incognitæ quantitatis (x & y) sit numerus correspondentium radicum respective $n \times \frac{n+3}{2}$; substituantur hæ correspondentes radices pro suis respectivis valoribus (x & y), & resultant æquationes.

$$\begin{aligned} \alpha^n + a + b\pi \times \alpha^{n-1} + c + d\pi + e\pi^2 \times \alpha^{n-2} + f + g\pi + h\pi^2 + k\pi^3 \times \alpha^{n-3} + \&c. &= 0, \\ \beta^n + a + b\rho \times \beta^{n-1} + c + d\rho + e\rho^2 \times \beta^{n-2} + f + g\rho + h\rho^2 + k\rho^3 \times \beta^{n-3} + \&c. &= 0, \\ \gamma^n + a + b\sigma \times \gamma^{n-1} + c + d\sigma + e\sigma^2 \times \gamma^{n-2} + f + g\sigma + h\sigma^2 + k\sigma^3 \times \gamma^{n-3} + \&c. &= 0, \\ \delta^n + a + b\tau \times \delta^{n-1} + c + d\tau + e\tau^2 \times \delta^{n-2} + f + g\tau + h\tau^2 + k\tau^3 \times \delta^{n-3} + \&c. &= 0, \\ \&c. \end{aligned}$$

Harum æquationum numerus per hypothefin æqualis est fractioni $n \times \frac{n+3}{2}$, & numerus incognitarum coefficientium ($a, b, c, d, e, \&c.$)

eidem quantitati æqualis est; reducantur hæ $\left(n \times \frac{n+3}{2}\right)$ æquationes in unam, ita ut exterminentur omnes præter unam (a) incognitæ coefficientes, & exinde constat e quot & qualibus membris, i. e. ex qua functione correspondentium radicum incognitarum quantatum ista una coefficiens constat; & sic de reliquis.

Cor. 1. Sit numerus correspondentium radicum incognitarum quantatum (x & y) æquationis, cujus dimensiones sunt (n), major quam $n \times \frac{n+3}{2}$, & e quibusque $\left(n \times \frac{n+3}{2}\right)$ correspondentibus incognitarum quantatum (x & y) radicibus oritur æquationis constitutio, i. e. coefficientes quæsitæ æquationis.

2. Sit numerus (m) correspondentium radicum minor quam $\left(n \times \frac{n+3}{2}\right)$, & infinitis modis dari possunt diversæ constitutiones æquationis, vel pro $\left(n \times \frac{n+3}{2} - m\right)$ coefficientibus scribi possunt quælibet quantitates.

Cor. 3. Sint duæ æquationes duas incognitas quantitates x & y habentes,

habentes, quarum dimensiones sunt n & m respective, & fit n minor quam m ; tum æquationis n dimensionum ita assumi possunt coefficientes utriusque æquationis, ut quicunque $\left(n \times \frac{n+3}{2}\right)$ numeri sint radices incognitæ quantitatis (x), & quicunque etiam $\left(n \times \frac{n+3}{2}\right)$ numeri sint correspondentes radices incognitæ quantitatis (y), æquationis vero m dimensionum coefficientes infinitis modis variari possint. Sint n & m inter se æquales, tum haud assumi possunt coefficientes utriusque æquationis, ita ut quicunque $\left(n \times \frac{n+3}{2}\right)$ numeri sint radices incognitæ quantitatis (x), & quicunque etiam $\left(n \times \frac{n+3}{2}\right)$ numeri sint correspondentes incognitæ quantitatis (y): si enim hoc fit, utraque æquatio eadem fiet: sed assumi possunt coefficientes infinitis modis, ita ut quicunque $\left(n \times \frac{n+3}{2} - 1\right)$ numeri sint respective correspondentes valores incognitarum quantitatum (x & y).

Ex. Sint duæ quadraticæ æquationes, duas incognitas quantitates habentes $y^2 + a + bx + y + c + dx + ex^2 = 0$ & $y^2 + A + Bx + y + C + Dx + Ex^2 = 0$.

in utrâque æquatione sunt quinque $\left(2 \times \frac{2+3}{2}\right)$ incognitæ coefficientes, substituantur pro incognitis quantitibus (x & y) quæcunque quinque quantitates respective in utrâque æquatione, & resultabunt ex utrâque eadem quinque simplices æquationes; unde constat utrasque æquationes datas necessario easdem resultare: ergo solummodo substitui possunt in utrâque æquatione quatuor correspondentes valores incognitarum quantitatum (x & y), quibus substitutis resultant ex utrâque quadraticâ quatuor simplices æquationes quinque incognitas coefficientes habentes, & consequenter in utrâque pro una incognita coefficiente assumi potest quicunque numerus; at cavendum est, ne idem numerus pro eadem coefficiente in utrisque æquationibus assumatur.

Sint quatuor valores incognitæ quantitatis (y), 0, 2, 4, 2; quatuor correspondentes valores incognitæ quantitatis (x) respective 0, 1, 1, 2; quibus substitutis & in utrâque quadraticâ pro unâ coefficiente (b & B) assumptis respective numeris 4 & -2 ; resultant e primâ quadraticâ quatuor simplices æquationes $c=0$, $4+2a+8+e+d=0$, $16+4a+16+e+d=0$, $4+2a+16+4e+2d=0$, e quibus invenietur prima æquatio $y^2 + 4x - 10 \times y - 8x^2 + 16x = 0$; & eodem modo invenitur secunda $y^2 - 2x + 4 \times y - 2x^2 + 10x = 0$.

Numerus correspondentium radicum in omni casu incognitarum quantitatum (x & y), quæ generaliter assumi possunt, æqualis est numero incognitarum datæ æquationis coefficientium.

Eadem est methodus ratiocinandi de pluribus æquationibus plures incognitas quantitates habentibus, sed observandum est æquationem

n dimensionum m incognitas quantitates involventem, $\frac{n+1}{n+1} \times \frac{n+2}{2} \times \frac{n+3}{3} \times \frac{n+m}{m} - 1$. incognitas coefficientes habere posse.

Facile constabit ex his æquationibus infinitas dari posse æquationum constitutiones.

P R O B. XXIX.

Sint duæ æquationes, duas incognitas quantitates (x & y) habentes, sint $\alpha, \beta, \gamma, \delta, \epsilon$, &c. radices incognitæ quantitatis (y), & $\pi, \rho, \sigma, \tau, \upsilon$, &c. correspondentes radices incognitæ quantitatis (x), & m & a dati indices; invenire summam.

$$\alpha^m \pi^a + \beta^m \rho^a + \gamma^m \sigma^a + \delta^m \tau^a + \&c.$$

Supponatur $y^m \times x^a = z$; & in datis æquationibus pro (y) substituatur ejus valor $\left(\frac{z^{\frac{1}{m}}}{x^{\frac{a}{m}}}\right)$; & transformentur duæ æquationes resultantes in unam, ut exterminetur incognita quantitas (x); & resultabit æquatio, cujus radix est z , & cujus radicum summa erit summa quæsitæ.

Cor. 2. Hæc radicum summa sæpe facilius investigari possit: & haud raro pendet e terminis datæ æquationis, quorum dimensiones sunt maximæ: & sic deinceps.

Prob.

PROB. XXX.

Iisdem literis easdem quantitates denotantibus, & positis m, n, o, r, s, &c. & a, b, c, d, e, &c. datis indicibus, invenire summam e singulis hujusmodi contentis

$\alpha^m \pi^a \beta^n \rho^b \gamma^o \sigma^c \&c + \alpha^m \pi^a \beta^n \rho^b \delta^o \tau^c \&c + \beta^m \rho^a \alpha^n \pi^b \gamma^o \sigma^c \&c + \&c.$
scribatur

$$p^m Q^a = \alpha^m \pi^a + \beta^m \rho^a + \gamma^m \sigma^a + \delta^m \tau^a + \epsilon^m \upsilon^a + \&c.$$

$$p^n Q^b = \alpha^n \pi^b + \beta^n \rho^b + \gamma^n \sigma^b + \delta^n \tau^b + \epsilon^n \upsilon^b + \&c.$$

$$p^o Q^c = \alpha^o \pi^c + \beta^o \rho^c + \gamma^o \sigma^c + \delta^o \tau^c + \epsilon^o \upsilon^c + \&c.$$

$$p^r Q^d = \alpha^r \pi^d + \beta^r \rho^d + \gamma^r \sigma^d + \delta^r \tau^d + \epsilon^r \upsilon^d + \&c.$$

$$\&c. \quad \&c. \quad \&c.$$

$$p^{m+n} Q^{a+b} = \alpha^{m+n} \pi^{a+b} + \beta^{m+n} \rho^{a+b} + \gamma^{m+n} \sigma^{a+b} + \&c.$$

$$p^{m+o} Q^{a+c} = \alpha^{m+o} \pi^{a+c} + \beta^{m+o} \rho^{a+c} + \gamma^{m+o} \sigma^{a+c} + \&c.$$

$$p^{n+o} Q^{b+c} = \alpha^{n+o} \pi^{b+c} + \beta^{n+o} \rho^{b+c} + \gamma^{n+o} \sigma^{b+c} + \&c.$$

$$p^{m+r} Q^{a+d} = \alpha^{m+r} \pi^{a+d} + \beta^{m+r} \rho^{a+d} + \gamma^{m+r} \sigma^{a+d} + \&c.$$

$$\&c. \quad \&c.$$

$$p^{m+n+o} Q^{a+b+c} = \alpha^{m+n+o} \pi^{a+b+c} + \beta^{m+n+o} \rho^{a+b+c} + \gamma^{m+n+o} \sigma^{a+b+c} + \&c.$$

$$p^{m+n+r} Q^{a+b+d} = \alpha^{m+n+r} \pi^{a+b+d} + \beta^{m+n+r} \rho^{a+b+d} + \gamma^{m+n+r} \sigma^{a+b+d} + \&c.$$

$$p^{n+o+r} Q^{b+c+d} = \alpha^{n+o+r} \pi^{b+c+d} + \beta^{n+o+r} \rho^{b+c+d} + \gamma^{n+o+r} \sigma^{b+c+d} + \&c.$$

$$\&c. \quad \&c.$$

$$p^{m+n+o+r} Q^{a+b+c+d} = \alpha^{m+n+o+r} \pi^{a+b+c+d} + \beta^{m+n+o+r} \rho^{a+b+c+d} + \gamma^{m+n+o+r} \sigma^{a+b+c+d} + \&c.$$

$$p^{m+n+o+s} Q^{a+b+c+e} = \alpha^{m+n+o+s} \pi^{a+b+c+e} + \beta^{m+n+o+s} \rho^{a+b+c+e} + \gamma^{m+n+o+s} \sigma^{a+b+c+e} + \&c.$$

$$\&c. \quad \&c.$$

Et sic progredi licet, observatâ problematis tertii substitutionis lege.
Scribatur etiam

$$A = P^m Q^a \times P^n Q^b \times P^o Q^c \times P^r Q^d \times \&c.$$

$$B = \frac{P^{m+n} Q^{a+b} \times A}{P^m Q^a \times P^n Q^b} + \frac{P^{m+o} Q^{a+c} \times A}{P^m Q^a \times P^o Q^c} + \frac{P^{n+o} Q^{b+c} \times A}{P^n Q^b \times P^o Q^c} + \&c.$$

$$C = \frac{P^{m+n+o} Q^{a+b+c} \times A}{P^m Q^a \times P^n Q^b \times P^o Q^c} + \frac{P^{m+n+r} Q^{a+b+d} \times A}{P^m Q^a \times P^n Q^b \times P^r Q^d} + \frac{P^{n+o+r} Q^{b+c+d} \times A}{P^n Q^b \times P^o Q^c \times P^r Q^d} + \&c.$$

$$D =$$

$$D = \frac{P^{m+n+r} Q^{a+b+c+d} \times A}{P^m Q^a \times P^n Q^b \times P^r Q^c \times P^d Q^d} + \frac{P^{m+n+r} Q^{a+b+c+d} \times A}{P^m Q^a \times P^n Q^b \times P^r Q^c \times P^d Q^d} + \frac{P^{m+n+r} Q^{a+b+c+d} \times A}{P^m Q^a \times P^n Q^b \times P^r Q^c \times P^d Q^d} + \&c.$$

$$BB = \frac{P^{m+n} Q^{a+b} \times P^{r+s} Q^{c+d} \times A}{P^m Q^a \times P^n Q^b \times P^r Q^c \times P^s Q^d} + \frac{P^{m+n} Q^{a+b} \times P^{r+s} Q^{c+d} \times A}{P^m Q^a \times P^n Q^b \times P^r Q^c \times P^s Q^d} + \&c.$$

observatâ problematis prædicti substitutionis lege; & summa quæsitâ erit

$$A - B + 2C - 1 \times 2 \times 3 D + 1 \times 2 \times 3 \times 4 E - \&c.$$

$$+ 1 \times 1 \times B B - 1 \times 1 \times 2 B C + \&c.$$

Eadem erit hæc ac problematis tertij series.

Observatis problematibus tertio & quarto, exinde constabit methodus inveniendi aggregatum e singulis valoribus cujuscunque functionis aut rationalis aut irrationalis radicum incognitarum quantitatum (x & y), etiamque infinitam radicum potestatum seriem.

Sæpe vero facilius e coefficientibus progredi liceat, & facile etiam inveniri possunt maximæ dimensiones, ad quas affurgat quæcunque incognita quantitas in resultanti quantitate.

Multæ etiam hujusmodi propositiones inveniri possunt e transpositione terminorum in alteram datæ æquationis partem; & addendo, subtrahendo, ducendo datas æquationes in se invicem, &c.

Cor. Sint plures æquationes, quæ tot incognitas quantitates ($x, y, z, v, \&c.$) habent; sint $\alpha, \beta, \gamma, \delta, \&c.$ radices incognitæ quantitatis (x); $\pi, \rho, \sigma, \tau, \&c.$ respective radices correspondentes quantitatis (y); $\kappa, \lambda, \mu, \nu, \&c.$ correspondentes radices incognitæ quantitatis (z); $\Gamma, \Delta, E, Z, \&c.$ correspondentes incognitæ quantitatis (v) valores, &c. sint etiam $a, f, l, p, \&c.$ dati indices; invenire summam $\alpha^a \pi^f \kappa^l \Gamma^p \&c. + \beta^a \rho^f \lambda^l \Delta^p \&c. + \gamma^a \sigma^f \mu^l E^p \&c. + \delta^a \tau^f \nu^l Z^p \&c. + \&c.$

Supponatur $x^a y^f z^l v^p \&c. = w$; & transformentur hæ æquationes resultantes in unam, ita ut exterminentur omnes incognitæ quantitates ($x, y, z, v, \&c.$) præter assumptam (w); & resultat æquatio, cujus radix est (w), & cujus radicum summa est summa quæsitâ.

Hoc etiam aliis methodis sæpe perfici possit.

Et

Et exinde per seriem, quæ eandem observat legem, quam observat præcedens hujus problematis series, investigari possit aggregatum e singulis valoribus cujuscunque functionis, aut rationalis, aut irrationalis radicum incognitarum quantitatuum ($x, y, z, v, \&c.$).

Et idem dici possit de aggregatis functionum quantitatuum prædictarum adventitiarum & radicum incognitarum quantitatuum ($x, y, z, v, \&c.$) conjunctarum inveniendis.

Sit α vera radix incognitæ quantitatis (x), substituatur hæc radix (α) in omnibus æquationibus pro ejus valore (x), & per radices adventitias eas intelligo, quæ per hanc substitutionem in quasdam æquationes irrepsent, omnibusque æquationibus non sunt communes.

P R O B. XXXI.

Transformare duas æquationes duas incognitas quantitates (x & y) habentes in unam, cujus incognita quantitas quamcunque habet algebraicam relationem ad incognitarum duarum æquationum quantitatuum radices.

1. Assumantur radices incognitarum quantitatuum tanquam cognitæ, i. e. sint radices incognitæ quantitatis (x) respective $\alpha, \beta, \gamma, \delta, \&c.$ & correspondentes radices incognitæ quantitatis (y) respective $\pi, \rho, \sigma, \tau, \&c.$ & ex datâ relatione (si modo haud per æquationes exprimatur relatio) inveniantur radices quæsitæ quantitatis; & consequenter per lemma præcedens summa ejus radicum, quæ est coefficientis secundi æquationis termini.

Ducantur quæque duæ, tres, &c. radices quæsitæ quantitatis in sese, & duorum præcedentium problematum ope inveniri possunt resultantium contentorum aggregata, quæ sint respective coefficientes tertii, quarti, &c. quæsitæ æquationis terminorum; & unde sequuntur quæsitæ æquationis coefficientes, & consequenter æquatio ipsa.

2. Si vero per æquationes quasunque datas relationem inter radices incognitarum datarum æquationum quantitatuum & quasunque assumptas incognitas quantitates exprimatur relatio.

Assumantur radices incognitarum quantitatuum datarum & æquationum

tionum tanquam cognitæ, & exinde inveniantur singulæ diversæ æquationes relationem exprimentes, multiplicentur omnes valores ex singulis hisce æquationibus continuo respective in se invicem; & e duobus problematibus præcedentibus inveniantur aggregata harum æquationum resultantium, quæ minime diversas ejusdem incognitæ quantitatis radices continebunt, & hæc aggregata erunt æquationes quæsitæ.

Cor. 1. Sint tres vel plures æquationes tres vel plures incognitas quantitates habentes, & eodem modo inveniri possint æquationes, quarum radices quamcunque habeant algebraicam relationem ad radices datarum æquationum.

Cor. 2. Si relatio inter incognitas datarum & quæsitæ æquationum quantitates solummodo correspondentes valores incognitarum quantitatum involvat; tum semper inveniri possunt quæsitæ æquationes e reductione plurium æquationes in unam, ita ut incognitæ quantitates exterminentur.

P R O B. XXXII.

Datis duabus vel pluribus æquationibus duas vel plures incognitas quantitates habentibus, invenire illas incognitarum quantitatum radices, quæ datam inter se habeant relationem.

1. Sint radices incognitarum quantitatum (x , y , &c.) sigillatim datæ.

Substituantur earum valores pro incognitis quantitativibus (x , y , &c.) & per methodum communes divisores inveniendi deduci possint radices, si modo quædam sint, quæsitæ.

Ex. Sint duæ æquationes

$$y^n + a + bx y^{n-1} + c + dx + ex^2 \times y^{n-2} + \&c. = 0,$$

$$\& y^m + A + Bx \times y^{m-1} + C + Dx + Ex^2 \times y^{m-2} + \&c. = 0,$$

invenire utrum sint tres radices incognitæ quantitatis (y) in arithmetica progressionem.

Sit

Sit arithmetica progressio $\alpha, \alpha + \beta, \alpha + 2\beta$; quibus valoribus in datis æquationibus pro suo valore (y) substitutis, resultant sex æquationes quinque solummodo incognitas quantitates habentes. E : G. Sint correspondentes valores incognitæ quantitatibus (x) respective π, ρ, σ ; & sex æquationes resultantes erunt

$$\begin{aligned} \alpha^n + a + b\pi \times \alpha^{n-1} + c + d\pi + e\pi^2 \times \alpha^{n-2} + \&c. = 0; \alpha + \beta + a + b\rho \times \alpha + \beta + c + d\rho + e\rho^2 \times \alpha + \beta + \&c. = 0; \\ \alpha^m + A + B\pi \times \alpha^{m-1} + C + D\pi + E\pi^2 \times \alpha^{m-2} + \&c. = 0; \alpha + \beta + A + B\rho \times \alpha + \beta + C + D\rho + E\rho^2 \times \alpha + \beta + \&c. = 0; \\ \alpha + 2\beta + a + b\sigma \times \alpha + 2\beta + c + d\sigma + e\sigma^2 \times \alpha + 2\beta + \&c. = 0, \\ \& \alpha + 2\beta + A + B\sigma \times \alpha + 2\beta + C + D\sigma + E\sigma^2 \times \alpha + 2\beta + \&c. = 0. \end{aligned}$$

Quarum quinque incognitæ quantitates erunt $\alpha, \beta, \pi, \rho, \sigma$: inveniantur, utrum hæ æquationes quosdam admittant valores, necne; & fit exemplum.

Ex. 2. Invenire, utrum ulla sit radix incognitæ quantitatibus (x), quæ æqualis est correspondenti alterius incognitæ quantitatibus (y) radici.

Sit radix quæsitæ (α); in datis æquationibus pro x & y scribatur (α), & inveniantur, utrum resultantes æquationes quosdam valores admittant, necne.

Ex. 3. Sint duæ æquationes $a + bx + cy + dx^2 + exy + fy^2 + gx^3 + bx^2y + kxy^2 + ly^3 + \&c. = 0$, & $A + Bx + Cy + Dx^2 + Exy + Fy^2 + Gx^3 + Hx^2y + Kxy^2 + Ly^3 + \&c. = 0$, invenire, utrum quidam sint correspondentes valores incognitarum quantitatibus x & y nihilo æquales: evanescant a & A termini utriusque æquationis, in quibus nulla invenitur dimensio, & erit correspondens valor incognitarum quantitatibus (x & y) nihilo respective æqualis; evanescant prædicti termini A & a , termini etiamque bx & cy & Bx & Cy ; & quatuor valores incognitæ quantitatibus (x) nihilo erunt æquales, hi autem quatuor valores quatuor etiam habebunt valores incognitæ quantitatibus (y) nihilo æquales: Evanescant termini $a, A, bx, cy, Bx, Cy, dx^2, exy, fy^2, Dx^2, Exy, Fy^2$; & novem erunt correspondentes valores incognitarum quantitatibus (x & y) nihilo æquales: & sic deinceps.

T

Sint

Sint termini ($r, r-1, r-2, r-3 \dots 0$) dimensionum unius æquationis, termini vero ($s, s-1, s-2, s-3 \dots 0$) dimensionum alterius, &c. nihilo æquales; & erunt ($s \times r$) radices resultantis æquationis nihilo æquales.

Sin vero termini, quorum dimensiones sint minimæ, habeant divisorem (x & y), tum quoad terminos minimarum dimensionum quam plurimi correspondentes valores nihilo fiant æquales; pendet e terminis, quorum dimensiones sunt proxime majores, &c. & sic deinceps.

Ex. 4. Sint duæ æquationes prædictæ; invenire utrum in genere (x) habeat valorem nihilo æqualem, necne; pro (x) in utrisque æquationibus scribatur (0), & inveniatur, utrum æquationes resultantes communem habeant divisorem, necne; si habeant divisorem, tot valores nihilo æquales habet incognita quantitas, quot dimensiones habet communis divisor.

Ex. 5. Sint duæ æquationes $y^{2n} + bx y^{2n-1} + \overline{c+ex^2} y^{2n-2} + \overline{gx+kx^3} y^{2n-3} + \&c. = A$, & $y^{2m} - bx y^{2m-1} + \overline{c+ex^2} y^{2m-2} - \overline{gx+kx^3} y^{2m-3} + \&c. = B$; vel sint $y^{2n} + \overline{c+ex^2} y^{2n-2} + \overline{l+mx^2+nx^4} y^{2n-4} + \&c. = 0$, & $y^{2m} + \overline{C+Ex^2} y^{2m-2} + \overline{L+Mx^2+Nx^4} y^{2m-4} + \&c. = 0$: & negativi & affirmativi valores incognitarum quantitatum (x & y) erunt inter se æquales.

Ex. 6. Datis duabus æquationibus $v y^n + \overline{a+bx} y^{n-1} + \overline{c+dx+ex^2} y^{n-2} + \overline{f+gx+bx^2+kx^3} y^{n-3} + \&c. = 0$, $N y^m + \overline{A+Bx} y^{m-1} + \overline{C+Dx+Ex^2} y^{m-2} + \overline{F+Gx+Hx^2+Kx^3} y^{m-3} + \&c. = 0$, duas incognitas quantitates (x & y) habentibus; invenire, utrum duæ radices incognitæ quantitatis (x) sint æquales, & duæ etiam correspondentes radices incognitæ quantitatis (y) æquales.

Ducantur termini successivi incognitarum quantitatum (x vel y) harum æquationum in terminos successivos arithmeticæ seriei; & resultant quantitates, quæ nihilo fiant æquales, cum duo valores incognitarum quantitatum (x & y) fiant respective inter se æquales, i. e. erunt

$nvy^{n-1} + \overline{n-1} \times \overline{a+bx} y^{n-2} + \overline{n-2} \times \overline{c+dx+ex^2} y^{n-3} + \&c. = 0$, &
 $mNy^{m-1} + \overline{m-1} \times \overline{A+Bx} y^{m-2} + \overline{m-2} \times \overline{C+Dx+Ex^2} y^{m-3} + \&c. = 0$, etiamque
 $by^{n-1} + \overline{d+2ex} y^{n-2} + \overline{g+2bx+3kx^2} y^{n-3} + \&c. = 0$, &
 $By^{m-1} + \overline{D+2Ex} y^{m-2} + \overline{G+2Hx+3Kx^2} y^{m-3} + \&c. = 0$, & præterea
 $\overline{n-1} by^{n-2} + \overline{n-2} d + 2\overline{n-2} ex y^{n-3} + \&c. = 0$, $\overline{m-1} By^{m-2} +$
 $\overline{m-2} D + 2\overline{m-2} Ex y^{m-3} + \&c. = 0$, cum duo valores incognitæ
 quantitates (x) fiant æquales, etiamque duo correspondentes valores
 incognitæ quantitatis (y) fiant æquales.

Cum vero radices haud sint æquales, limites inter radices inveni-
 ant hæ resultantes æquationes.

Ex. 7. Sint duæ æquationes $y^n + \overline{a+bx} y^{n-1} + \overline{c+dx+ex^2} y^{n-2}$
 $+ \&c. + Ax^m + Bx^{m-1} + Cx^{m-2} + Dx^{m-3} + \&c. = 0$, & $y^n + \overline{a+bx}$
 $y^{n-1} + \overline{c+dx+ex^2} y^{n-2} + \&c. + Px^r + Qx^{r-1} + Rx^{r-2} + \&c.$
 $= 0$, i. e. omnes termini, in quibus continetur (y), sint iidem in u-
 trâque æquatione; & (n) erunt valores incognitæ quantitatis (x) sem-
 per inter se æquales.

1. Si relatio exprimatur per æquationes: e reductione plurium æ-
 quationum in unam, ita ut incognitæ quantitates exterminentur, in-
 veniri possunt illæ radices, si modo quædam sint.

2. Si vero data sit æquatio, quæ exprimat relationem inter diver-
 fas ejusdem incognitæ quantitatis (x) radices, tum ita reducantur
 datæ æquationes ut exterminentur omnes præter incognitam quanti-
 tatem (x), & inveniatur utrum data æquatio sit divisor æquationis re-
 sultantis, necne.

3. Datis quibusdam functionibus incognitarum quantitatum radi-
 cum, e datis æquationibus inveniuntur hæ functiones, & e resultan-
 tibus & datis æquationibus inveniuntur radices incognitarum quanti-
 tatum quæsitæ, si modo ullæ sint.

Cor. Sint duæ vel plures æquationes, quæ duas vel plures incogni-
 tas quantitates ($x, y, z, v, \&c.$) habent, & eadem inveniatur irratio-
 nalitas

nalitas in correspondentibus incognitarum quantitatum ($x, y, z, v, \&c.$) valoribus; ni duo vel plures valores incognitæ quantitatis (x) sint inter se æquales.

Cor. 1. Datâ quâcunque quantitate e diversis irrationalibus quantitibus ($A + B + C + D + E + \&c.$) conflata, invenire, utrum ullos recipiat possibiles valores, necne.

Assumatur pro irrationalis quantitatis A radice $a + p\sqrt{-1}$, irrationalis quantitatis B radice $b + q\sqrt{-1}$, irrationalis quantitatis C radice $c + r\sqrt{-1}$, irrationalis quantitatis D radice $d + s\sqrt{-1}$, $\&c.$

Supponatur $p + q + r + s + \&c. = 0$, & per problema xxx inveniatur, utrum æquationes resultantes ex assumptis valoribus talem habeant relationem ($p + q + r + s + \&c. = 0$), necne.

Cor. 2. Datis duabus vel pluribus æquationibus duas vel plures incognitas quantitates ($x, y, z, \&c.$) habentibus, & sit impossibilis quantitas $a + \sqrt{-b^2}$ radix incognitæ quantitatis (x), & $a - \sqrt{-b^2}$ etiam erit radix ejusdem incognitæ quantitatis (x).

Constat enim e substitutione horum valorum pro incognitâ quantitate (x).

Haud abs re forsan erit sequentia adjungere.

Datis duabus æquationibus duas incognitas quantitates (x & y) habentibus, & si radix ($a + \sqrt{-b^2}$) incognitæ quantitatis (x) sit impossibilis, tum ejus correspondens valor incognitæ quantitatis (y) etiam erit impossibilis, ni duo, vel quatuor vel sex, $\&c.$ valores incognitæ quantitatis (y) sint inter se æquales, qui correspondent prædictæ radici incognitæ quantitatis (x). Et sic de pluribus incognitis quantitibus in pluribus æquationibus contentis.

Sint (n) æquationes (n) incognitas quantitates ($x, y, v, w, z, \&c.$) habentes; invenire, quot sint possibiles valores correspondentium radicum ($x, y, z, v, \&c.$) inveniatur quot possibiles valores habet quâcunque incognita quantitas (x), & tot erunt possibiles correspondentes valores e singulis incognitis quantitibus ($y, z, v, \&c.$), ni duo vel quatuor vel plures valores incognitæ quantitatis (x) sint inter se æquales;

æquales; in quo casu inveniatur e præcedente problemate, utrum possibiles valores incognitarum quantitatum ($y, z, v, \&c.$) respondeant duobus vel pluribus æqualibus valoribus incognitæ quantitatis (x), necne: & id, quod quæsitum est, perficitur.

Si hæ æquationes constant e diversis quadratis, tum vel omnes radices sint impossibiles, vel duo vel plures valores incognitarum quantitatum sint inter se æquales, et facile exinde inveniri possint infinitæ æquationes, quæ nullos habent possibiles radices.

1^{mo}. E substitutione quantitatum pro radicibus in datis æquationibus; 2^{do}, e quantitibus e datis æquationibus deductis; 3^{tio}, e transformatione æquationum datarum in alias, quarum radices assignabilem habent relationem ad radices datarum æquationum; 4^{to}, ex inveniundo, quando duæ radices cujuscunque incognitæ quantitatis semel, bis, ter, &c. fiant æquales, deduci possunt regulæ impossibiles datarum æquationum radices detegendi: transformentur datæ æquationes in alias, quarum radices assignabilem habeant relationem ad radices datarum æquationum, & e numero impossibilium radicum resultantium æquationum inveniri possit numerus impossibilium radicum datarum æquationum, & vice versâ.

Hæc etiam vel consimilia applicari possunt in numero affirmatarum & negativarum radicum incognitarum quantitatum investigando.

Sint duæ æquationes duas incognitas quantitates (x & y) habentes, invenire quot respondentes valores incognitarum quantitatum (x & y) eadem habeant signa, quot autem diversa.

Inveniatur æquatio, cujus radix fit $\frac{x}{y}$, & quot ejus radices sint affirmativæ, tot signa respondentium incognitarum quantitatum (x & y) erunt eadem; quot autem negativæ, tot erunt diversa: ni ulla radix impossibilis incognitæ quantitatis (x) fit $a + b\sqrt{-1}$, ejus vero correspondens radix incognitæ quantitatis (y) fit $ma + mb\sqrt{-1}$.

Facile etiam inveniri possint æquationes, quarum radices sint omnes negativæ, vel omnes affirmativæ.

Omnia

Omnia ea, quæ prius docuimus de impossibilibus radicibus unius æquationis, unam incognitam quantitatem habentis, etiam applicari possunt ad impossibiles radices duas vel plures incognitas quantitates habentes. Et sic de affirmativis & negativis radicibus.

Omnis æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, (n) dimensionum erit differentia inter duas æquationes easdem (n) vel plures dimensiones habentes; & quarum omnes radices sint possibiles, vel m radices sint possibiles & affirmativæ, (r) negativæ, cæteræ vero impossibiles; & quæ æquationes infinitis modis variari possint.

Quædam ex his etiam in summam duarum vel plurium æquationum (n) dimensionum applicari possint.

Et consimilia applicari possint ad duas vel plures æquationes incognitas quantitates habentes.

PROB. XXXIII.

Datis duabus æquationibus $A = 0$, & $P = 0$ duas incognitas quantitates (x & y) habentibus, & divisores recipiant hæ æquationes, i. e. sint $A = \alpha \times \beta \times \gamma \times \&c.$ & $P = \pi \times \rho \times \sigma \times \tau \times \&c.$ deprimere datas æquationes in alias minores habentes dimensiones.

Supponantur singuli unius datæ æquationis divisores, singuli vero alterius nihilo æquales, & exinde resultant æquationes reductæ, i. e. erunt æquationes reductæ $\alpha = 0$, & $\pi = 0$; $\alpha = 0$, & $\rho = 0$; $\alpha = 0$, & $\sigma = 0$; $\alpha = 0$, & $\tau = 0$; &c. $\beta = 0$, & $\pi = 0$; $\beta = 0$, & $\rho = 0$; $\beta = 0$, & $\sigma = 0$; $\beta = 0$, & $\tau = 0$; &c. $\gamma = 0$, & $\pi = 0$; $\gamma = 0$, & $\rho = 0$; $\gamma = 0$, & $\sigma = 0$; $\gamma = 0$, & $\tau = 0$; &c. $\delta = 0$, & $\pi = 0$, &c. utrum divisores admittant datæ æquationes, e subsequenti capite inveniri potest.

Cor. Sint duæ æquationes $y^n + \overline{a+bx} y^{n-1} + \&c. = 0$, $y^m + \overline{A+Bx} y^{m-1} + \&c. = 0$, quarum divisores sint respective $y + \alpha x + \beta$, $y + \gamma x + \delta$, $y + \epsilon x + \zeta$, &c. & $y + \pi x + \rho$, $y + \sigma x + \tau$, $y + \upsilon x + \mu$, &c. Sint prioris æquationis divisores (n), posterioris vero (m). Supponantur singuli unius æquationis divisores, & singuli alterius nihilo

hilo æquales, i. e. $y + \alpha x + \beta = 0$, & $y + \pi x + \rho = 0$ (e quibus æquationibus colligitur $x = \frac{\rho - \beta}{\alpha - \pi}$, & $y = \frac{\pi \beta - \alpha \rho}{\alpha - \pi}$): $y + \alpha x + \beta = 0$,

& $y + \sigma x + \tau = 0$, (ex hisce æquationibus colligitur $x = \frac{\tau - \beta}{\alpha - \sigma}$,

& $y = \frac{\sigma \beta - \tau \alpha}{\alpha - \sigma}$), &c. & sic deinceps; unde $n \times m$ erunt diversi correspondentes valores incognitarum quantitatum (x & y), fed minime ($n \times m$) assumi possunt generaliter correspondentes valores; quot autem assumi possunt, ut prædictum est, pendet e numero incognitarum coefficientium: correspondentes valores quendam habeant nexum inter se, i. e. omnes diversi valores ex unâ datâ æquatione resultantes comparantur cum omnibus diversis valoribus ex alterâ resultantibus; datis igitur quibusdam valoribus, necessario orientur reliquæ.

Idem dici possit de æquationibus divisores haud recipientibus, hoc enim in casu singuli diversi valores ejusdem incognitæ quantitatis ex unâ datâ æquatione vel exprimi possunt per irrationales quantitates vel per infinitas series terminorum, hi autem valores comparari possint cum valoribus ex alterâ æquatione similiter resultantibus.

Cor. Sint duæ æquationes (n) dimensionum, duas incognitas quantitates habentes, & prius docetur haud generaliter assumi posse $n \times \frac{n+3}{2}$ diversos correspondentes valores e duabus incognitis quantitibus, aliter utraque æquatio eadem fiet: quodsi ita assumantur correspondentes valores prædictam habentes affinitatem inter se, tum n^2 correspondentes valores assumi possint, etiamque n^2 diversæ erunt æquationes, fed $n^2 + 1$ vero diversi correspondentes valores probant vel utramque æquationem eandem esse, vel communem inter se divisorem habere.

Sint duæ æquationes, quæ divisores haud admittant; ducantur hæ æquationes in assumptas quantitates, & æquationum resultantium inveniantur summa vel differentia, & nonnunquam resultabunt æquationes, quæ divisores recipiant: utrum divisores admittant resultantes æquationes, e subsequenti capite colligi potest.

Ex.

Ex. Sint duæ æquationes, $6y^2 - 5y - 4x^2 + 34x - 31 = 0$,
 & $y^2 - 11x - 1y + 14x^2 - 9x + 4 = 0$; hæ æquationes autem
 nullum recipient divisorem; subtrahatur posterior æquatio e bis priori,
 & dividatur residuum per 11, & resultat æquatio $y^2 + x - 1y - 2x^2$
 $+ 7x - 6 = y - 1x + 2xy + 2x - 3 = 0$.

Addantur prior æquatio & quinquies secunda, & dividatur summa
 per 11, & resultat æquatio $y^2 - 5xy + 6x^2 - x - 1 = y - 2x + 1$
 $\times y - 3x - 1 = 0$; unde correspondentes radices incognitarum quanti-
 tatum (x & y) erunt respectivæ $-1, 1, -\frac{3}{2}, \frac{2}{5}$; & $-3, 1, -\frac{7}{2},$
 $\frac{11}{5}$.

Hinc facile inveniri possunt infinitæ hujusmodi æquationes. As-
 sumantur enim duæ æquationes, quæ divisores admittant, & quæ
 nullos habent communes inter se divisores; ducantur hæ æquationes
 in quasdam assumptas quantitates, & e resultantium æquationum sum-
 mis, &c. constari possunt æquationes, quæ nullos admittant divi-
 fores.

Et sic de pluribus æquationibus plures incognitas quantitates ha-
 bentibus.

Reduci possint æquationes, ita ut incognitæ quantitates extermin-
 entur, & inveniri possint æquationum divisores ita dividendo æqua-
 tionem hâc methodo deductam in duas partes, ut ex utrâque parte
 eadem radix extrahatur: etiamque hæc perfici possunt e transforma-
 tione datarum æquationum in alias, quarum radices datam habeant
 assignabilem relationem ad radices datarum æquationum, hâc etiam
 methodo tolli possunt ex hisce æquationibus duo vel plures termini
 æque ac ex æquatione unam solummodo incognitam quantitatem ha-
 bente: et inveniri possunt æquationes, quas reducere licet; transfor-
 mentur enim datæ æquationes, ita ut resultantes plures habeant di-
 mensiones quam datæ, & rursus reducantur transformatæ æquationes
 in datas, & sic deinceps. Et ex eodem modo, quo prius docetur in-
 vestigandi constitutionem transformatæ æquationis unam solummodo
 incog-

incognitam quantitatem habentis methodus, investigari etiam possunt coefficientes transformatarum æquationum plures incognitas quantitates habentium.

Invenire æquationes, quas resolvere licet.

Assumantur (n) resolutiones pro singulis (n) incognitis quantitatibus resolvendis; & ita reducantur, ut exterminentur irrationales quantitates, & ex æquationibus resultantibus formari possunt (n) æquationes (n) incognitas quantitates habentes.

Invenire æquationes, quæ deprimi possunt: assumantur æquationes, ita transformentur hæ æquationes ut plures habeant dimensiones æquationes transformatæ quam datæ, & transformatæ facile deprimi possunt in datas.

Hæ methodi haud multum diversæ sunt ab iis in capite præcedenti contentis, in quibus tractatum est de unâ æquatione unam solummodo incognitam quantitatem habente.

Haud nunquam in perfectas potestates reduci possint datæ æquationes, & exinde radices extrahendo reducentur æquationes.

Hæ potestates semper facile perfectæ reddantur, si modo fieri possint, ex additione vel subductione, &c.

Haftenus de methodis deprimendi datas æquationes duas vel plures incognitas quantitates habentes in alias, vel minores dimensiones habentes, vel quarum formulæ vel sint minores vel generis, cujus resolutio vel reductio cognoscitur.

Si vero ita transformentur datæ æquationes in alias, ut transformatæ æquationes easdem habeant dimensiones, quas habeant datæ; & termini duarum vel plurium æquationum, in quibus dimensiones incognitarum quantitatum inveniantur maximæ, communem habeant divisorem; vel evanescant termini, in quibus nullæ inveniuntur dimensiones incognitarum quantitatum, vel magis generaliter quædam radices nihilo fiant æquales, vel ita transformentur datæ æquationes in alias, quarum dimensiones eadem sunt ac datæ, & quædam radices resultantium æquationum æquales inter se fiant, tum deprimantur dimensiones datarum æquationum incognitarum quantitatum.

Utiq. in his aliis æquationibus facile reduci possunt. Ex. I.

Ex. 1. Sint æquationes $x^n + n \times \frac{n-1}{2} x^{n-2} y^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} x^{n-4} y^4 + \&c. = A$, & $x^{n-1} y + \frac{n-1}{2} \times \frac{n-2}{3} \times x^{n-3} y^3 + \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times x^{n-5} y^5 + \&c. = B$, & erit $x + y = \sqrt[n]{A + nB}$, & $x - y = \sqrt[n]{A - nB}$: erunt enim (n) valores summæ $(x + y)$ & differentiæ $(x - y)$ semper inter se æquales, & consequenter æquatio, cujus radix fit $x + y$ vel $x - y$ haud plures habet quam n dimensiones; æquatio autem, cujus radix fit x vel y habet $n \times n$ dimensiones.

Ex. 2. Sint æquationes $x + y \times x^{n-1} - a x^{n-2} y + \frac{n-2}{2} \times \frac{n-1}{2} x^{n-3} y^2 - \frac{n-2}{2} \times \frac{n-3}{3} a x^{n-4} y^3 + \&c. = A$, & $x + y \times x^{n-2} y + \frac{n-2}{2} \times \frac{n-3}{3} x^{n-4} y^3 + \&c. = B$, & erit $x + y = \sqrt[n]{A + a + n-1B}$.

Ex. 3. Sint $x - y \times x^{n-1} - a x^{n-2} y + \frac{n-2}{2} \times \frac{n-1}{2} x^{n-3} y^2 - \frac{n-3}{2} \times \frac{n-2}{3} a x^{n-4} y^3 + \&c. = A$, & $x - y \times x^{n-2} y + \frac{n-3}{2} \times \frac{n-2}{3} x^{n-4} y^3 + \&c. = B$, & erit $x - y = \sqrt[n]{A + a - n-1B}$.

Et sic de infinitis hujuscemodi æquationibus.

Ex. 4. Sint datæ æquationes $x^m y^r z^s v^t \&c. = A$, $x^i y^j z^k v^l \&c. = B$, $x^{ii} y^{jj} z^{kk} v^{ll} \&c. = C$, $x^{iii} y^{jjj} z^{kkk} v^{lll} \&c. = D$, &c.

Facile constat æquationes, quarum radices vel sint x , vel y , vel z , vel v , &c. esse quoad earum formulas simplices: & ex his additione vel subtractione, &c. infinitæ formari possunt æquationes, quæ retro in has æquationes facile reduci possint.

THEOR.

THEOR. VII.

Sint duæ vel plures æquationes duas incognitas quantitates (x & y) habentes & similiter involventes, reducantur hæ æquationes in unam, ita ut exterminetur altera incognita quantitas (y), & æquatio resultans, cujus radix est x , duplas habet dimensiones æquationis, cujus radix est $x + y$, vel $x^2 + y^2$, vel $x^3 + y^3$, vel xy , &c. *demonstratio.* Sint radices incognitæ quantitatis (x) respectivæ $\alpha, \beta, \gamma, \delta, \dots, \pi, \rho, \sigma, \tau$; correspondentes vero radices incognitæ quantitatis (y) erunt respectivæ $\tau, \sigma, \rho, \pi, \dots, \delta, \gamma, \beta, \alpha$; & incognitæ quantitatis ($z = x + y$) respectivi valores sunt $\alpha + \tau, \beta + \sigma, \gamma + \rho, \delta + \pi, \dots, \pi + \delta, \gamma + \rho, \beta + \sigma, \alpha + \tau$, unde omnis valor incognitæ quantitatis ($z = x + y$) alterum habet sibi ipsi æqualem ($\alpha + \tau, \alpha + \tau, \beta + \sigma, \beta + \sigma$, &c.), & consequenter incognita quantitas (x aut y) duplum habet dimensionum numerum, quem habet incognita quantitas ($x + y$); & æquatio, cujus radix est (x aut y), duplas habet dimensiones æquationis, cujus radix est $x + y$, &c.

Ex. 1. Sint duæ æquationes $\overline{x^2 + y^2} \times \overline{x^2 + y^2} = b$, & $\overline{x^2 + y^2} \times \overline{x + y} = a$; utriusque æquationis terminorum, in quibus incognitarum quantitarum (x vel y) dimensiones sunt maximæ, communis est divisor ($x^2 + y^2$); unde sequitur æquationem, cujus radix est (x vel y) dimensiones ($6 \times 3 = 18$) non habere, ejus dimensiones, ut facile constat, erunt (12); sed hæ duæ æquationes incognitas quantitates (x & y) similiter involvunt, & consequenter æquatio, cujus radix est ($z = x + y$) solummodo sex habet dimensiones, & facile patet ejus formulam esse quadraticam, viz. $az^6 + 2b - 2a^2z^3 - a^3 = 0$.

Ex. 2. Sint duæ æquationes $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = a$ & $x^4 + x^3y + x^2y^2 + xy^3 + y^4 \times x^4 - x^3y + x^2y^2 - xy^3 + y^4 = b$; harum æquationum termini, in quibus incognitarum quantitarum (x aut y) dimensiones sunt maximæ, communem habent divisorem ($x^4 + x^3y + x^2y^2 + xy^3 + y^4$); & consequenter æquatio, cujus radix est (x aut y), non habet ($4 \times 8 = 32$) dimensiones; habet vero (4×4)

= 16) dimensiones; & eadem, ut e prædictis constat, erit ejus formula ac octo dimensionum æquatio, i. e. ejus negativæ & affirmativæ radices erunt inter se æquales: sed hæ duæ æquationes incognitas quantitates (x & y) similiter involvunt; & exinde æquatio, cujus radix est ($z = x^2 + y^2$), non habere potest plures quam $\left(\frac{16}{4} = 4\right)$ dimensiones; etiamque constat ejus formulam esse quadraticam, viz.

$$4a^2 z^2 - 2a^2 + 2b \times a z^2 = a^2 - b.$$

Ex. 3. Sint duæ æquationes $x^4 + y^4 \times x^2 + y^2 = bx^2 y^2$, & $x^2 + y^2 \times x + y = axy$; & quoniam in his duabus æquationibus ($x^2 + y^2$) erit communis divisor terminorum, in quibus maximæ inveniuntur (sex & tres) dimensiones; & duodecim radices nihilo inveniuntur æquales, ex eo, quod evanescant termini unius æquationis, in quibus nulla vel una inveniatur dimensio; alterius vero evanescant termini in quibus inveniuntur nulla, una, duæ, & tres dimensiones; & terminorum, duarum dimensionum unius æquationis, & quatuor dimensionum alterius (xy) est divisor; ergo æquatio, cujus radix fit x vel y quatuor solummodo habet dimensiones; & exinde, quoniam similiter involvuntur termini x & y , æquatio, cujus radix fit $x + y = z$, duas solummodo habebit dimensiones, erit enim $z^2 + \frac{b}{a}z + \frac{b-a^2}{2} = 0$.

Ex. 4. Sint duæ æquationes $x^3 + x^2 y + x y^2 + y^3 = axy$, & $x^3 + x^2 y + x y^2 + y^3 \times x^6 - x^5 y + x^3 y^3 - x y^5 + y^6 = bx^3 y^3$, & constat e prædictis rationibus æquationem, cujus radix fit x vel y habere posse $9 \times 3 - 9 \times 2 - 3 = 6$ dimensiones; & consequenter æquatio, cujus radix fit $x + y = z$, habet tres dimensiones, & erit $z^3 - az^2 - \frac{a^2}{3} + \frac{2b}{3a}z + \frac{a^3}{3} - \frac{b}{3} = 0$.

Ex. 5. Sint duæ æquationes $x^3 + x^2 y + x y^2 + y^3 = axy$, & $x^{3n} + x^{2n} y^n + x^n y^{2n} + y^{3n} = bx^n y^n$; fit $1^{\text{mo}} n = 6m + 3$ & æquatio, cujus radix fit x vel y , habet $3n \times 3 - 3n \times 2 - 3 = 3n - 3 = 18m + 6$ dimensiones; & æquatio, cujus radix fit $x + y$, habet

$9m + 3$ dimensiones: si vero $n = 2m$; tum æquatio cujus radix fit x vel y , habet $3n \times 3 - 3 \times n \times 2 - 2 = 3n - 2 = 6m - 2$ dimensiones; & æquatio, cujus radix fit $x + y$ habet $3m - 1$ dimensiones: fit $n = 6m \pm 1$; & æquatio, cujus radix fit x vel y habet $3n - 1$ dimensiones; & æquatio, cujus radix fit $x + y$ habet $\frac{3n - 1}{2} = \frac{18m \pm 3 - 1}{2} = 9m - 2$ vel $9m + 1$; ubi m semper denotat integrum numerum.

Et in genere æquatio, cujus radix fit $z = x + y$, erit $z^n + a - z^n - \frac{nz^3}{2z+a} \times z^{n-2} + a - z + n \times \frac{n-3}{2} \times \frac{z^6}{2z+a} \times z^{n-4} + a - z + \&c.$
 $= b$: lex hujusce seriei constabit e problemate primo.

Ex. 6. Sint duæ æquationes $x^2 + zx + z^2 + zy + y^2 = az$, ubi $z^2 = xy$, & $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = bxy$, & facile constat æquationem, cujus radix fit z , duas solummodo habere

dimensiones: erit $z^2 + \frac{1}{2}a + \frac{b}{2a} \times z - \frac{a}{2} - \frac{b}{2a} = 0$.

Æquatio autem $v = \frac{a}{2} - \frac{b}{2a}$, cujus radix $v = x + y$ duas habet radices æquales.

Et sic sint duæ æquationes $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5 = ax^2y^2$, & $x^{10} + x^8y^2 + x^6y^4 + x^4y^6 + x^2y^8 + y^{10} = bx^4y^4$; & æquatio, cujus radix fit x vel y , habet $5 \times 10 - 5 \times 8 = 10$ dimensiones; æquatio autem cujus radix fit $x + y$ habet $\frac{10}{2} = 5$ dimensiones.

Sint etiam duæ æquationes $x^7 + x^6y + x^5y^2 + x^4y^3 + x^3y^4 + x^2y^5 + xy^6 + y^7 = ax^3y^3$, & $x^{14} + x^{12}y^2 + x^{10}y^4 + x^8y^6 + x^6y^8 + x^4y^{10} + x^2y^{12} + y^{14} = bx^6y^6$; & æquatio, cujus radix fit x vel y , habet 14 dimensiones; ergo æquatio, cujus radix fit $x + y$, habet 7 dimensiones: Et sic in infinitas consimiles æquationes progredi licet; sed de his satis.

THEOR.

THEOR. VIII.

Sit quicunque æquationum numerus (n) tot (n) incognitas quantitates habentium, & sint $(x, y, v, w, \&c.)$ quædam incognitæ quantitates similiter inter se in singulâ æquatione involutæ, & quarum numerus sit (m) ; sint etiam $(X, Y, V, W, \&c.)$ incognitæ quantitates similiter involutæ inter se in singulâ æquatione, & quarum numerus sit (r) ; sint etiam $(\alpha, \beta, \gamma, \delta, \&c.)$ quantitates incognitæ similiter in singulâ æquatione inter se involutæ, & quarum numerus sit (s) ; & sic deinceps: & numerus dimensionum æquationis, cujus radix sit $x + y + z + v + \&c. + X + Y + Z + V + \&c. + \alpha + \beta + \gamma + \delta + \&c.$ erit ad numerum dimensionum æquationis, cujus radix sit $x + y + \&c. + X + Y + \&c. + \alpha + \beta + \&c. + \&c.$ (in cujus summa tot sint literæ $x, y, \&c.$ quot unitates in m ; tot sint literæ $X, Y, \&c.$ quot unitates in r ; tot sint literæ $\alpha, \beta, \&c.$

quot unitates in s ; &c.) :: $1 : m \times \frac{m-1}{2} \times \frac{m-2}{3} \dots \frac{m-m+1}{m} \times r \times$

$\frac{r-1}{2} \times \frac{r-2}{3} \dots \frac{r-r+1}{r} \times s \times \frac{s-1}{2} \times \frac{s-2}{3} \dots \frac{s-s+1}{s}$; si vero m sit 0,

tum pro $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \dots \frac{m-m+1}{m}$ scribatur 1: & sic de

reliquis.

Ex. Sit numerus (n) incognitarum quantitarum $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ & sint (n) datæ æquationes $\alpha + \beta + \gamma + \delta + \epsilon + \&c. = a$, $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \&c. = b$, $\alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \&c. = c$, $\alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \&c. = d$, &c. Et dimensiones æquationis, cujus radix sit $\alpha + \beta + \gamma + \delta + \&c. = a$ erunt ad dimensiones æquationis, cujus radix sit $\alpha :: 1 : n$. ergo æquatio, cujus radix sit α , n habet dimen-

siones, etiamque erit $\alpha^n - a\alpha^{n-1} + \frac{a^2-b}{2}\alpha^{n-2} - \frac{a^3-3ab+2c}{6}$

$\alpha^{n-3} + \frac{a^4-6a^2b+8ac-6d}{24}\alpha^{n-4} - \&c. = 0$. Lex hujusce seriei

constat e corollario ad problema tertium.

Et

Et sic de infinitis hujusmodi exemplis. Facile enim fingi possunt infinita hujusmodi exempla.

Cor. Eadem eruat dimensiones æquationis, cujus radices sint quæcunque rationalis quantitas, in quâ literæ x, y, v, w , &c. & X, Y, V, W , &c. & $\alpha, \beta, \gamma, \delta$, &c. &c. similiter involvantur.

SCHOLIUM.

In radicibus duarum vel plurium æquationum duas vel plures incognitas quantitates (x, y , &c.) habentium investigandis, frequentissime esset frustra per prædictas methodos progredi; confugiendum enim est ad continuas approximationes.

Sint e. g. duæ datæ æquationes duas incognitas quantitates (x & y) habentes $y^n + a + bx y^{n-1} + \&c. = 0$, & $y^m + A + Bx \times y^{m-1} + \&c. = 0$, & sint correspondentes radices incognitarum quantitarum (x & y) respective $\alpha, \beta, \gamma, \delta$, &c. & π, ρ, σ, τ , &c. invenire ex approximatione correspondentes valores (α & π) ex incognitis quantitatibus (x & y); necesse est, ut prius dentur quantitates (k & l) ad prædictos correspondentes valores (α & π) multo magis appropinquantes, quam ad ullum alium e correspondentibus incognitarum quantitarum (x & y) radicibus (β, γ, δ , &c. & ρ, σ, τ , &c.): In datis æquationibus pro quantitatibus (x & y) scribantur respective $z - k$ & $v - l$, & sint æquationes resultantes $\Gamma + \Delta z + E v + Z z^2 + H z v + \Theta v^2 + \&c. = 0$, & $\Pi + P z + \Sigma v + T z^2 + \Upsilon z v + \Phi v^2 + \&c. = 0$; & e constitutione coefficientium colligi possint $\Gamma + \Delta z + E v$ prope $= 0$, & $\Pi + P z + \Sigma v$ prope $= 0$, & exinde approximatio ad z erit $\frac{E\P - \Sigma\Gamma}{\Sigma\Delta - E\P}$

& approximatio ad v erit $\frac{\Delta\P - P\Gamma}{P E - \Delta\Sigma}$.

Si vero k & l multo magis approximant ad duas correspondentes radices incognitarum quantitarum (x & y) respective, quam ad ullas alias; & pro x & y substituantur respective $z - k$ & $v - l$ in datis æquationibus; & e constitutione coefficientium colligi possint vel $\Gamma + \Delta z + E v$ prope $= 0$, & $\Pi + P z + \Sigma v + T z^2 + \Upsilon z v + \Phi v^2$ prope

prope $= 0$; vel $\Pi + Pz + \Sigma v$ prope $= 0$, & $\Gamma + \Delta z + Ev + Zz^2 + Hzv + \Theta v^2$ prope $= 0$, e quibus æquationibus inveniri possunt approximationes ad quantitates z & v : Et sic progredi licet, cum k & l ad tres vel plures correspondentes incognitarum quantitatum radices multo magis approximent, quam ad reliquas.

Etiamque facile constat methodus inveniendi rationem, quam habet approximatio ad veram radicem, a terminis diversarum radicum incognitarum quantitatum expressam.

Invenire autem continuas approximationes ad impossibilem radicem $(a + b\sqrt{-1})$ in unâ æquatione unam solummodo incognitam quantitatem, vel ad impossibiles correspondentes incognitarum quantitatum radices in duabus vel pluribus æquationibus duas vel plures incognitas quantitates habentibus, necesse est, ut haud solummodo dentur quantitates magis appropinquantes ad possibiles partes radicum quæsitæ respectivè, quam ad possibiles partes reliquarum incognitarum quantitatum radicum, sed etiam dentur quantitates ad impossibiles partes quæsitæ radicum magis appropinquantes quam ad impossibiles partes reliquarum incognitarum quantitatum radicum: & ex eadem prorsus methodo, quæ in possibilibus radicibus investigandis prius docetur, erui possunt continuæ approximationes & ad possibiles & impossibiles quæsitæ radicum partes. Et omnia, quæ prius docentur de possibilibus radicibus, de impossibilibus etiam vere affirmari possint: omnia præcedentia æque in (n) æquationes (n) incognitas quantitates habentes applicari possint.

CAP. V.

DE RATIONALIBUS ET INTEGRIS
QUANTITATIBUS.

PROB. XXXIV.

*D*atâ æquatione unam solummodo incognitam quantitatem habente
 $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} \dots Px^3 + Qx^2 + Rx + S = 0$,
 invenire ejus rationales divisores.

1^{mo}. Sint $p, q, r, \&c. P, Q, R, S$ integri numeri; & quoniam
 $x^{n-1} + px^{n-2} + qx^{n-3} + rx^{n-4} \dots Px^2 + Qx + R \times x = -S$,
 radix (x), si modo sit integer numerus, erit divisor ultimæ datæ æqua-
 tionis coefficientis (S).

Transformetur data æquatio in alteram, cujus radices vel majores
 vel minores sint quam radices datæ æquationis per datum numerum
 (a); & $(x - a)$ erit divisor ultimi resultantis æquationis termini $a^n +$
 $pa^{n-1} + qa^{n-2} + ra^{n-3} + \&c.$

E: G Sint a successive 0, 1, -1, 10, -10, 100, -100, &c. & z
 radix datæ æquationis; & $z, z-1, z+1, z-10, z+10, z-100,$
 $z+100, \&c.$ erunt respectivi divisores ultimorum resultantium æ-
 quationum terminorum; i. e. substituantur pro radice x in datâ æqua-
 tione 0, 1, -1, 10, -10, &c. & numerorum resultantium $z, z-1,$
 $z+1, z-10, z+10, \&c.$ erunt respectivi divisores.

Sint a respective 0, 1, -1, 2, -2, 3, -3, &c. & z radix datæ æ-
 quationis; & $z, z-1, z+1, z-2, z+2, z-3, z+3, \&c.$ erunt
 divisores ultimorum resultantium æquationum terminorum, i. e. nu-
 merorum resultantium e substitutione quantitatum 0, 1, -1, 2, -2,
 3, -3, &c. pro radice (x) in datâ æquatione.

E prædictis divisoribus igitur inveniri possint datæ æquationis ratio-
 nales radices.

Hoc problema invenit Jacobus a Wassenæ.

X

2^{do}. Si

2^{do}. Si vero fractionales quantitates in datâ æquatione contineantur, mutari potest in aliam, quæ nullam contineat fractionalem quantitatem, e : g. fit æquatio $Nx^n + px^{n-1} + qx^{n-2} + rx^{n-3} + \&c. \dots Qx^2 + Rx + S = 0$: ubi $N, p, q, r, \&c. Q, R, S$ integros denotent numeros; scribatur $\frac{z}{N}$ pro x , & resultabit æquatio $z^n + pNz^{n-1} + qN^2z^{n-2} + rN^3z^{n-3} + \&c. \dots QN^{n-2}z^2 + RN^{n-1}z + SN^n = 0$, cujus radix fit (α) , & $Nx - \alpha$ erit divisor quæsitus.

Hinc omnis radix, quæ sit rationalis fractio, pro suo denominatore habebit aliquem numerum, qui sit divisor numeri N . Sit autem (α) divisor quantitatis S , & m divisor quantitatis N , & substituantur pro x in datâ æquatione $(Nx^n + px^{n-1} + qx^{n-2} + rx^{n-3} \dots Qx^2 + Rx + S = 0)$ $0, 1, -1, 10, -10, \&c.$ & quantitarum resultantium sint $\alpha, \alpha - m, \alpha + m, \alpha - 10m, \alpha + 10m, \&c.$ respectivi divisores: vel si modo substituantur pro x successive $0, 1, -1, 2, -2, \&c.$ in datâ æquatione, & quantitarum resultantium sint $\alpha, \alpha - m, \alpha + m, \alpha - 2m, \alpha + 2m, \&c.$ respectivi divisores; & $m\alpha - \alpha$ erit quantitas pro divi-fore datæ æquationis tentanda.

Eodem modo transformetur data æquatio in aliam, cujus radices sint quæcunque rationalis functio datæ æquationis radicum; ultimi, & datæ & resultantis æquationis termini sint α & π divisores respective; & π erit ea rationalis functio divisoris (α) ; si modo α sit rationalis radix quæsitâ.

Est etiam alia methodus inveniendi divisores æquationis $S + Rx + Qx^2 + Px^3 + Ox^4 \dots x^n = 0$, in quâ $S, R, Q, P, \&c.$ integri sint numeri.

Sit a radix quæsitâ, qui sit integer numerus; & $\frac{S}{a}, \frac{\frac{S}{a} + R}{a}, \frac{\frac{S}{a} + R}{a} + Q, \frac{\frac{S}{a} + R}{a} + Q + \frac{P}{a}, \&c.$ integri erunt numeri; i. e. si modo $A, B, C, D, E,$

E , &c. sint præcedentes quotientes, $\frac{S}{a}, \frac{A+R}{a}, \frac{B+Q}{a}, \frac{C+P}{a}, \frac{D+O}{a}$,
 &c. vel quod idem est $\frac{S}{a}, \frac{Ra+S}{a^2}, \frac{Qa^2+Ra+S}{a^3}, \frac{Pa^3+Qa^2+Ra+S}{a^4}$,
 &c. integri erunt numeri, e: g. fit data æquatio $-35 - 44x + 7x^2 - 7x^3 + x^4 = 0$; pro S, R, Q, P, O scribantur respective $-35, -44, 7, -7, 1$: cœfficientis (-35) divisores $1, 5, 7, -1, -5, -7$, &c.
 pro rationali radice (a) assumatur divisor 7 , & $\frac{S}{a} = -\frac{35}{7} = -5$;
 $\frac{A+R}{a} = \frac{-5-44}{7} = -7$; $\frac{B+Q}{a} = \frac{-7+7}{7} = 0$; $\frac{C+P}{a} = \frac{0-7}{7} = -1$;
 $\frac{D+O}{a} = \frac{-1+1}{7} = 0$; unde sequitur numerum 7 esse rationalem datæ æquationis divisorem.

2. Si vero fractionales quantitates habeat data æquatio, mutari potest in aliam, quæ nullam contineant fractionalem quantitatem; & deinde præcedenti methodo inveniri potest, utrum integros habeat valores, necne: aliter.

Sit cœfficiens termini (x^n) quicunque integer numerus (N), cujus divisor sit m , & $\left(\frac{a}{m}\right)$ quæsitæ radix, & datæ æquatio $S + Rx + Qx^2 + Px^3 + O x^4 \dots N x^n = 0$; tum

$$\frac{S}{a}, \frac{\frac{mS}{a} + R}{a}, \frac{m \times \frac{\frac{mS}{a} + R}{a} + Q}{a}, \dots$$

&c. vel si modo A, B, C, D , &c. præcedentes denotent quantitates,
 $\frac{S}{a}, \frac{mA+R}{a}, \frac{mB+Q}{a}, \frac{mC+P}{a}, \frac{mD+O}{a}$, &c. integri erunt numeri;
 vel quod idem est $\frac{S}{a}, \frac{Ra+mS}{a^2}, \frac{Qa^2+mRa+m^2S}{a^3}, \frac{Pa^3+mQa^2+m^2Ra+m^3S}{a^4}$, &c. integri erunt numeri.

3. Si nullus hâc methodo occurrat divisor, concludendum est æquationem haud admittere simplicem divisorem, potest tamen habere quadraticum divisorem, qui sit $mx^2 - ax + b$; in quo casu b erit divisor ultimi datæ æquationis termini (S), & m divisor coefficientis (N) datæ æquationis termini (x^n); & $m - a + b$ divisor quantitatis $S + R + Q + P + O + \&c.$ & $m + a + b$ divisor quantitatis $S - R + Q - P + O - \&c.$ etiamque

$$\frac{R + \frac{aS}{b}}{b} \left(\frac{Rb + aS}{b^2} \right), \frac{Q - \frac{mS}{b} + a \times \frac{R + \frac{aS}{b}}{b}}{b} \left(\frac{Qb^2 - mSb + bRa + a^2S}{b^3} \right),$$

&c. integri erunt numeri: vel si modo $A, B, C, D, E, \&c.$ respective denotent præcedentes numeros, tum $\frac{S}{b} (A), \frac{R + aA}{b} (B), \frac{Q + aB - mA}{b}$ (C), $\frac{P + aC - mB}{b} (D), \&c.$ integri erunt numeri.

Et sic lex facile constat, e quâ detegi possint trium vel plurium dimensionum divisores. Aliter vero inveniri possunt rationales quadratici datæ æquationis divisores ($mx^2 - ax + b$). In datâ æquatione pro radice (x) substitue tres vel quatuor vel plures terminos hujus progressionis 3, 2, 1, 0, -1, -2, -3, &c. divisores omnes numerorum resultantium figillatim adde & subduc quadratis correspondentium terminorum progressionis illius ductis in aliquem numeralem divisorem (m) quantitatis (N), & summas differentiasque e regione progressionis colloca. Dein progressionem omnes collaterales nota, quæ per istas summas differentiasque percurrunt: Sit $\pm b$ terminus istiusmodi progressionis primæ; $\pm a$ differentia, quæ oritur subducendo $m \pm b$ de termino maxime superiori, qui stat e regione termini (1) progressionis primæ.

Si vero nullum habet quadraticum divisorem data æquatio; potest tamen habere cubicum, quadrato-quadraticum, &c. qui inveniri possint, quærendo in prædictis summis differentiisque progressionem non arithmeticas quidem sed alias quasdam quarum terminorum differentia primæ, secundæ, tertiæ, &c. sunt in arithmetica progressionem.

Eodem

Eodem modo transformetur data æquatio, cujus radices sint quæcumque rationalis functio datæ æquationis radicum, & exinde inveniri possint rationales quadratici, cubici, &c. divisores quæfiti. Inveniendi divisorem quantitatis, quæ unam solummodo continet literam eadem erit methodus, ac æquationis unam solummodo incognitam quantitatem habentis.

4. Datâ quantitate duas vel plures incognitas quantitates involvente, invenire utrum ullum divisorem recipiat, necne.

1^{mo}. Supponantur omnes quantitates præter unam nihilo æquales, & inveniantur divisores quantitatum resultantium, deinde supponantur omnes præter duas incognitæ quantitates nihilo æquales, & inveniantur divisores quantitatum resultantium; & sic de reliquis: vel aliter, collocentur singuli datæ quantitatis termini juxta dimensiones cujuslibet literæ (y), & inveniantur divisores ejus ultimi termini; & substituantur pro (y) quilibet numeri -2 , -1 , 0 , $+1$, $+2$, &c. & exinde per præcedentes methodos inveniri possint divisores quæfiti.

Si plures sint quam duæ incognitæ quantitates, tum solummodo pluries repetenda est operatio: singula enim operatio unam destruet incognitam quantitatem.

2^{do}. Si vero fractiones contineantur in datâ quantitate; tum singuli termini datæ quantitatis reducendi sunt in communem denominatorem; & divisores & numeratoris & denominatoris quærendi sunt: si vero irrationales termini in datâ quantitate contineantur; pro singulis diversis irrationalibus terminis substitue diversas literas, & per præcedentem methodum inveniendi divisores quantitatis duas vel plures incognitas quantitates involventis erui possint divisores quæfiti.

Cor. In quibusdam casibus facile consequitur ex aliis observationibus æquationes haud admittere rationales divisores. E. g. Sit æquatio $x^{2n} + 3ax^n + 3bx^r + \&c. = 3d + 2$, ubi a , b , &c. d , integri sint numeri; tum quoniam nullus quadratus (x^{2n}) numerus per 3 divisus relinquit 2, æquatio haud admittet rationalem divisorem: & sic e residuis post divisionem per quoscunque numeros infinitæ consimiles propositiones inveniri possint, vel e limitibus inter quos necessario ponitur radix quæsitæ.

THEOR.

THEOR. X.

Sint duæ æquationes duas incognitas quantitates (x & y) habentes; sit x rationalis quantitas, & y etiam erit rationalis quantitas; & quantitates (x & y) semper eandem habebunt irrationalitatem; i. e. quantitas x per easdem irrationales quantitates exprimitur, ac quantitas (y), ni duo vel plures valores quantitatuum (x vel y) sint inter se æquales: si vero duo (α & β) sint diverfi valores quantitatibus (x), quæ correspondent uni valori (π) incognitæ quantitatibus (y); tum quadratica æquatio, cujus radices sint α & β nullas habet irrationales quantitates, quæ non inveniuntur in correspondente valore (π) incognitæ quantitatibus (y): si vero tres valores (α , β , γ) incognitæ quantitatibus (x) correspondent uni valori (π) incognitæ quantitatibus (y), tum cubica æquatio, cujus radices sint α , β , γ , nullas habet irrationales quantitates, quæ haud inveniuntur in (π): & sic deinceps.

Eadem affirmari possunt de pluribus æquationibus plures incognitas quantitates habentibus.

Cor. Sint duæ æquationes duas incognitas quantitates (x & y) habentes; & sint coefficientes terminorum, in quibus maximæ inveniuntur dimensiones incognitæ quantitatibus (x) in utrâque æquatione respective N & H ; coefficientes vero terminorum, in quibus maximæ inveniuntur dimensiones incognitæ quantitatibus (y) in utrâque æquatione respective S & s ; & si modo y & x sint rationales quantitates & y integer numerus, x erit fractio, cujus denominator erit communis divisor coefficientium N & H ; & vice versâ sit x integer numerus, & y erit fractio, cujus denominator erit communis divisor coefficientium S & s : Si vero quantitas (x) sit quæcunque fractio, cujus denominator sit α ; & maximæ dimensiones quantitatibus (x) in utrâque æquatione respective sint q & r ; tum quantitas (y) erit fractio, cujus denominator est divisor communis quantitatuum $\alpha^q S$ & $\alpha^r s$. Sint $S y^m + A x y^{m-1} + B x^2 y^{m-2} + \&c.$ & $s y^n + a x y^{n-1} + b x^2 y^{n-2} + \&c.$ termini respectivi duarum æquationum, in quibus maximæ inveniuntur dimensiones incognitarum quantitatuum (x & y); prioris quantitatibus sint $y - \alpha x$, $y - \beta x$, $y - \gamma x$, $y - \delta x$, &c. simplices divisores;

res; posterioris vero $y - \pi x, y - \rho x, y - \sigma x, \&c.$ & quantitas x erit fractio, cujus denominator est divisor contenti $S \times s \times \alpha - \pi \times \alpha - \rho \times \alpha - \sigma \times \alpha \&c. \times \beta - \pi \times \beta - \rho \times \beta - \sigma \times \beta \&c. \times \gamma - \pi \times \gamma - \rho \times \gamma - \sigma \times \gamma \&c.$

Eadem etiam facile applicari possint, cum nihilo sint æquales quicunque termini.

P R O B. XXXV.

Datis duabus æquationibus duas incognitas quantitates habentibus, invenire integrales correspondentes, si quas forte habeant, incognitarum quantitatuum (x & y) radices.

1. In datis æquationibus pro incognitâ quantitate (y) substitue tres vel plures terminos arithmeticæ seriei $-1, 0, 1, 2, \&c.$ vel seriei $-1, 0, 1, 10, \&c.$ duas quantitates resultantes ex singulâ substitutione colloca juxta dimensiones literæ (x), ita ut illi termini primum locum occupent, in quibus litera ista (x) est plurimarum dimensionum, &c. & perpetuâ ablatione minoris de majori, & reliqui de ablato, exterminetur incognita quantitas (x); & numerorum resultantium e priori substitutione sint termini arithmeticæ seriei $z+1, z, z-1, z-2, \&c.$ divisores respectivi: e posteriori sint termini $z+1, z, z-1, z-10, \&c.$ divisores respectivi: tum divisor z erit integralis incognitæ quantitatibus (y) quæsitâ radix.

Ex. Sint duæ æquationes $y^2 + 1 + 3x \times y + 2x^2 + 4x - 18 = 0$, & $y^2 + 3xy + 3x^2 + 4x - 17 = 0$; pro y substituatur (0), & resultant quantitates $2x^2 + 4x - 18$, & $3x^2 + 4x - 17$; transformentur hi termini, ita ut exterminetur (x), & operatio est

$$\begin{array}{r} 2x^2 + 4x - 18 \times 3 = 6x^2 + 12x - 54 \\ 3x^2 + 4x - 17 \times 2 = 6x^2 + 8x - 34 \text{ auferatur inferior de superiori, \& } \\ \text{residuum erit} \quad 4x - 20, \text{ tum } 2x^2 + 4x - 18 \times 2 = 4x^2 + 8x - 36 \\ 4x - 20 \times x = 4x^2 - 20x \\ \text{residuum} \quad 28x - 36 \\ 4x - 20 \times 7 = 28x - 140 \\ \text{residuum} \quad 104 \\ \text{cujus} \end{array}$$

cujus numeri (104) integralis radix quantitatis (y) erit divisor: Et sic substituantur 1 & — 1 pro quantitate (y), & numeri resultantes erunt 512 & 972: quorum numerorum 512, 104, 972, &c. inveniantur divisores, & unica solummodo est arithmetica progressio (1, 2, 3, &c.), cujus communis differentia est (1); ergo numerus (2), qui fuit divisor numeri 104 resultantis e substitutione (0) pro incognitâ quantitate (y) erit integralis quantitatis (y) radix.

E problematibus præcedentibus facile deduci potest methodus inveniendi fractionales correspondentes radices incognitarum quantitatum (x & y).

Eodem modo quadraticæ, cubicæ, &c. æquationes, quarum radices sint radices incognitæ quantitatis (x) duarum æquationum duas vel plures incognitas quantitates (x , &c.) habentium, inveniri possunt, si modo duo, tres, &c. correspondentes valores incognitæ quantitatis (y) sint æquales; sin aliter ad reductionem æquationum necessarium est confugere.

Et sic in pluribus similiter ac in unâ æquatione unam incognitam quantitatem habente: transformentur datæ æquationes in alias, quarum radices sint quæcunque rationales datarum æquationum radicum functiones; & exinde deduci possit methodus eruendi rationales datarum æquationum radices.

Nonnunquam facile constat, quod rationalis valor cujuscunque incognitæ quantitatis (x); i. e. ejus numerator & denominator vel sint divisores, &c. datorum numerorum, vel consistant inter quosdam limites, in quibus casibus forsan facile inveniri possint rationales valores quæsi.

2. Est etiam alia methodus hoc problema resolvendi: Sint duæ æquationes duas incognitas quantitates habentes $S - Ry + Qy^2 - Py^3 + \&c. \dots y^n = 0$, & $s - ry + qy^2 - py^3 + \&c. \dots y^m = 0$, in quibus S, R, Q, P , &c. s, r, q, p , &c. denotent respectivé integras functiones literæ (x), & sit z rationalis integra radix quantitatis (y) quæsi; tum $\frac{S}{z}, \frac{Rz - S}{z^2} \left(\frac{R - A}{z} \right), \frac{Qz^2 - Rz + S}{z^3} \left(\frac{Q - B}{z} \right),$
&c.

&c. $\frac{s}{z}, \frac{rz-s}{z^2} \left(\frac{r-a}{z} \right), \frac{qz^2-rz+s}{z^3} \left(\frac{q-b}{z} \right),$ &c. (ubi $A, B, C,$ &c. $a, b, c,$ &c. præcedentes denotent fractiones) integri erunt numeri.

Ea, quæ prius dicta fuerunt de fractionalibus radicibus, & quadraticis, cubicis, &c. æquationibus inveniendis, cujus radices sint datarum æquationum incognitæ quantitatis radices, huc etiam proferre liceat.

Cum vero requiratur, ut radices sint integri numeri: sæpe è limitibus, inter quos facile constat quasdam consistere radices, vel e divisoribus numerorum facile deduci possint incognitarum quantitatum integri valores.

P R O B. XXXVI.

Datâ æquatione, invenire, utrum resolvi potest extractionum radicum quadraticarum, cubicarum, &c. ope.

Datâ æquatione, dantur etiam ejus dimensiones; assumatur generaliter resolutio, quæ solummodo habet radices quadraticas, cubicas, &c. & ita ut, si modo exterminentur surdæ quantitates, resultet æquatio datarum dimensionum: Fiant resultans & data æquatio inter se æquales, & confit problema.

Ex. Invenire, utrum data quatuor vel octo dimensionum æquatio resolvi potest extractionum radicum quadraticarum ope, necne.

Assumatur generaliter resolutio, quæ radices habet quadraticas, ita ut resultans æquatio quatuor vel octo habeat dimensiones, & erit re-

solutio $x = \sqrt[2]{a+m\sqrt{b}+n\sqrt{c}},$ vel $x = \sqrt[2]{a+m\sqrt{b+n\sqrt{c}+o\sqrt{c}}}$
 $+ r\sqrt{b+n\sqrt{c}} + s\sqrt{c} + d:$ reducantur hæ æquationes, ita ut exterminentur irrationales quantitates, & resultant æquationes $x^4 -$
 $4cx^3 - 2n^2b - 6c^2 + 2ax^2 - 4c^3 - 4ac + 4mnb - 4n^2cbx +$
 $n^2b + c^2 - a - m - 2nc \times b = 0,$ & $x^8 - 8dx^7 + \&c. = 0.$ Fi-
 ant resultantes & datæ æquationes inter se æquales, & exinde tot re-
 sultant æquationes, quot incognitæ quantitates; unde e præcedenti-

Y

bus

bus problematibus inveniri possint, utrum quantitates a, m, b, n, c vel $a, m, b, n, c, o, r, s, d$ integros habeant valores, necne; & confit problema.

Et methodo haud dissimili de æquationibus plures dimensiones habentibus, & extractionibus ope quadraticarum, cubicarum, &c. radicum agere liceat.

PROB. XXXVII.

Reducere æquationem $x^{2n} + px^{2n-1} + qx^{2n-2} + rx^{2n-3} + sx^{2n-4} + tx^{2n-5} + \&c. = 0$ *in duas alias* $x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + dx^{n-4} + ex^{n-5} + fx^{n-6} + \&c. = \sqrt{l} \times \alpha x^{n-1} + \beta x^{n-2} + \gamma x^{n-3} + \delta x^{n-4} + \&c.$ *extractionis quadraticæ radicis ope.*

Reducatur æquatio, & resultat $x^{2n} + 2ax^{2n-1} + 2b + a^2 - l\alpha^2 \times x^{2n-2} + 2c + 2ba - 2l\alpha\beta x^{2n-3} + 2d + 2ac + b^2 - l \times 2\alpha\gamma + \beta^2 x^{2n-4} + 2e + 2da + 2bc - l \times 2\alpha\delta + 2\beta\gamma x^{2n-5} + \&c. = 0$: Fiant respectivi datæ & resultantis æquationis termini inter se æquales, & resultant $(2n)$ æquationes $2a = p, 2b + a^2 - l\alpha^2 = q, 2c + 2ab - 2l\alpha\beta = r, 2d + 2ac + b^2 - l \times 2\alpha\gamma + \beta^2 = s, 2e + 2ad + 2bc - l \times 2\alpha\delta + 2\beta\gamma = t, \&c.$ Hæ autem $2n$ æquationes habent $(2n + 1)$ incognitas quantitates, sed harum incognitarum una (l) semper ducta fuit in (n) incognitas quantitates ($\alpha, \beta, \gamma, \delta, \&c.$); reducantur igitur hæ $(2n)$ æquationes in (n) æquationes, ita ut exterminentur (n) incognitæ quantitates, ($a, b, c, \&c.$) & resultant (n) æquationes nullas incognitas quantitates habentes, præter eas ($\alpha, \beta, \gamma, \delta, \epsilon, \&c.$), in quas incognita quantitas (l) semper ducta fuit: ergo resultant (n) numeri, quorum omnium l est divisor: i. e. e primâ æquatione $a = \frac{p}{2}$; e secundâ (si modo $\pi = q -$

$\frac{1}{4} p^2$) $2b - l\alpha^2 = \pi$; e tertiâ (si modo $\varrho = r - \frac{1}{2} p \pi$) $2c = \varrho + l \times 2\alpha\beta - \frac{p\alpha^2}{2}$; e quartâ (si modo $\sigma = s - \frac{1}{2} p \varrho - \frac{1}{4} \pi^2$) $2d = \sigma - l \times p\alpha\beta - \frac{p^2\alpha^2 - 2\pi\alpha^2 - l\alpha^4}{4} - 2\alpha\gamma - \beta^2$; & sic deinceps: & quæ-

rendus

rendus est communis divisor integer & non quadratus (n) numero-
 rum resultantium, qui per 4 divisus relinquat unitatem, si modo ter-
 minorum alternorum p, r, t , &c. aliquis sit impar: Per divisores in-
 telligo tales fractiones, quales habent potestates numeri 2 pro suis de-
 nominatoribus: si a sit dimidium imparis numeri, tum (α) etiam erit
 dimidium imparis numeri; si b , tum β ; si c , tum γ ; & sic deinceps:
 Si vero communem divisorem (l) admittant prædicti (n) numeri;
 quære numerum quadratum (Γ^2), cui per (l) multiplicato ultimus
 æquationis terminus sub signo proprio annexus quadratum numerum
 efficit: sit enim data æquatio $P + Qx + Rx^2 + \&c. = 0$; assumatur
 æquatio $A + Bx + Cx^2 + \&c. = \sqrt{l} \times \Gamma + \Delta x + Ex^2 + \&c.$ &
 resultat data æquatio $A^2 - l \times \Gamma^2 + 2AB - \Gamma\Delta x + \&c. = 0$, un-
 de $P = A^2 - l \Gamma^2$, $Q = 2AB - 2\Gamma\Delta$, &c.

Hæc limitationes admittant ex hâc observatione, quod $A \pm \sqrt{l} \Gamma$,
 $B \pm \sqrt{l} \Delta$, $C \pm \sqrt{l} E$, &c. haud majores erunt quantitates quam
 quadruplus maximus terminus datæ æquationis, vel ex infinitis aliis.

Cor. Sit $x^4 + px^3 + qx^2 + rx + s = 0$, & supponatur $x^2 + \frac{p}{2}x$
 $+ b = \sqrt{l} \times \alpha x + \beta$: sit $q - \frac{1}{4}p^2 = \pi$, & $r - \frac{1}{2}p\pi = \rho$, $s -$
 $\frac{1}{4}\pi^2 = \sigma$, & e prædictâ regulâ constat l esse communem divisorem
 terminorum ρ & 2σ , & non quadratum, & per (4) divisum unitatem
 relinquere, si terminorum p & r alteruter sit impar: pone pro α divi-
 forem aliquem quantitatis $\frac{\rho}{l}$, si p sit par; vel imparis divisoris dimi-
 dium, si p sit impar; aufer quotum de $\frac{1}{2}p\alpha$, & reliqui dimidium dic
 β ; deinde pro b pone $\frac{\pi + l\alpha^2}{2}$; & tenta, si l dividat $b^2 - s$; & quoti
 radix sit rationalis, & æqualis l ; & si hoc fit, reducitur data æquatio,
 sin aliter reductionem haud admittet.

Omnia hæc e præcedentibus æquationibus, viz. ($a = \frac{p}{2}$, $2b = \pi + l\alpha^2$, $l \times \frac{p\alpha^2}{2} - 2\alpha\beta = q$, & $s = b^2 - l\beta^2$) conjunctim consideratis facile sequuntur.

Alia deduci possit methodus hoc corollarium resolvendi; deleatur secundus datæ æquationis terminus, & resultet æquatio $x^4 + 2x^2 + Rx + S = 0$: inveniatur, utrum æquatio $P^6 + 2qP^4 + q^2 - 4sP^2 - r^2 = 0$ admittat rationalem radicem, & haud quadratum, & exinde constabit corollarium; multæ confimiles methodi in æquationes majorum dimensionum facile deduci possint.

Cor. 2. Datis n æquationibus numerum (m), qui minor sit quam n , incognitarum quantitatuum habentibus, & quæ haud continuo ducantur in incognitam quantitatem (l), & quoque numerum (r) incognitarum quantitatuum, quæ continuo ducantur in incognitam quantitatem (l), & ita reducantur (n) datæ æquationes, ut exterminentur (m) incognitæ quantitates, in quas haud continuo ducta fuit incognita quantitas (l), & exinde deduci possint ($n - m$) numeri, quorum (l) erit divisor communis.

PROB. XXXVIII.

Sit data æquatio $x^{4n} + px^{4n-1} + qx^{4n-2} + rx^{4n-3} + sx^{4n-4} + tx^{4n-5} + \&c. = 0$ dimensionum ($4n$), invenire utrum deprimi possit in æquationem (n) dimensionum radicum quadraticarum extractionum ope.

Assumatur æquatio $x^n + ax^{n-1} + bx^{n-2} + \&c. + \sqrt{l \times ax^{n-1} + \beta x^{n-2} + \&c.} = \sqrt{r + s \sqrt{l \times \pi x^{n-1} + q x^{n-2} + \&c.}}$ reducatur, ita ut exterminentur irrationales quantitates, & resultat

$$x^{4n} + 4ax^{4n-1} + 4b - 2r\pi^2 + 6a^2 - 2l\alpha^2 x^{4n-2} + \&c. = 0.$$

Hinc colliguntur æquationes $4a = p$, $4b - 2r\pi^2 + 6a^2 - 2l\alpha^2 = q$, &c. & resultant ($4n$) æquationes ($3n+3$) incognitas quantitates habentes, & quarum ($n+1$) incognitæ quantitates $\alpha, \beta, \&c. s$ ducantur in incognitam quantitatem (l); exterminentur omnes incognitæ quantitates

titates præter prædictas ($\alpha, \beta, \&c. s$) & resultabunt ($2n-1$) numeri, quorum quantitas (l) erit communis divisor. Omnis incognita quantitas vel erit integer numerus, vel impar numerus per duo vel quatuor, &c. divisus.

Hæ incognitæ quantitates inveniri possint e methodo communes divisores inveniendi, quoniam plures dantur æquationes quam incognitæ quantitates: vel aliter e principiis in priori problemate contentis.

Et sic in genere inveniri possit, utrum æquatio reduci potest extractionum radicum quadraticarum, vel cubicarum, &c. ope.

Etiamque eadem applicari possint in plures æquationes plures incognitas quantitates habentes.

P R O B. XXXIX.

Datâ æquatione invenire utrum ullas habeat radices datæ formulæ, in cujus formulâ literæ contentæ semper rationales quantitates denotent.

Per methodum in capite tertio traditam inveniatur, utrum ullæ sint radices, quæ datam habeant formulam, necne: & per proble. 34 & 35 inveniatur, utrum literæ contentæ sint integrales vel rationales, necne.

In quam plurimis casibus multo facilius erui possit solutio e terminis correspondentibus inter se comparandis.

Ex. 1. Sit formula $\sqrt{x} + \sqrt{y}$, ubi x & y rationales denotent quantitates, & x major sit quam y : & sit data æquatio $w^2 = a + \sqrt{b}$, & erit $x = \frac{a + \sqrt{a^2 - b}}{2}$, & $y = \frac{a - \sqrt{a^2 - b}}{2}$; & exinde constat ejus radices (w) haud datam habere formulam, ni $a^2 - b$ vel sit quadratum vel 0.

Sit $w^2 = \sqrt{a} + \sqrt{b}$, & facile reduci possit in formulam $w^2 = \sqrt{a} \times 1 + \sqrt{\frac{b}{a}}$ vel in formulam $w^2 = \sqrt{b} \times 1 + \sqrt{\frac{a}{b}}$.

Eadem methodus etiam applicari possit ad extrahendam radicem quatuor vel octo vel sedecim &c. dimensionum; vel ad extrahendam radicem quadraticam, &c. e tribus vel pluribus hujuscemodi quantitatibus.

tatibus $a + \sqrt{b} + \sqrt{c} + \sqrt{d} + \&c.$ vel citius inveniri possit radix e pluribus quam duabus quantitatibus dividendo factum quarumvis duarum radicalium per tertiam aliquam radicalem, quæ producit quotum rationalem & integrum; nam quoti istius radix erit duplum partis radicis quæsitæ.

Ex. 2. Sit quantitas $A \pm B = a + \sqrt{b}$, ejus major pars A , & index radicis extrahendæ n : Quære numerum minimum N , cujus potestas N^n dividitur per $A^n - B^n$ sine residuo, ut calculus ad minimos numeros redigatur, & sit quotus Q : computa $\sqrt[n]{A+B} \sqrt[n]{Q}$ in numeris integris proximis; fit illud r , divide $A \sqrt[n]{Q}$ per maximum rationalem

divisorem; fit quotus s , & sit $\frac{r + \frac{N}{r}}{2s}$ in integris numeris prox-

imis t ; & erit $\frac{ts + \sqrt{t^2 s^2 - N}}{\sqrt[n]{Q}}$ radix quæsitæ.

Cæterum in hujusmodi operationibus, si quantitas fractio sit, vel partes ejus communem habeant divisorem: radices denominatoris & factorum seorsim extrahe.

Hæc est regula Schootenii a Neutono concinne reddita.

Quidam autem sint casus, in quibus radices haud detegit hæc regula, ni prius transformetur data quantitas, cujus requiritur radix in alteram, quod primus animadvertit celeberrimus Euler, & pro prædictâ regulâ alteram substituit: Sit data quantitas $A + B$, cujus radix n potestatis requiritur: Sit radix quæsitæ $\frac{x+y}{\sqrt[n]{p}} = \sqrt[n]{A+B}$, unde $x^n - y^n =$

$\sqrt[n]{A^n - B^n} \times p$, ubi x, y & p sint integri numeri, & $x^n - y^n = A^n - B^n$ $\times p = r^n$, ubi r minimus sit numerus potestatis n , per quam dividi potest $A^n - B^n$, ut calculus ad minimos terminos redigatur: Cognitâ differentiâ quadratorum $x^2 - y^2 = r$; quære etiam summam quadratorum

torum $x^2 + y^2 = \frac{1}{2} \sqrt[n]{A+B \times p} + \frac{1}{2} \sqrt[n]{A-B \times p}$; si vero radix
 exhiberi possit, necesse est ut summa binarum irrationalium quantita-
 tum fiat integer numerus. Hic itaque prodibit, si numeri integri
 quærantur proxime accedentes ad valores irrationales $\sqrt[n]{A+B \times p}$
 & $\sqrt[n]{A-B \times p}$, fumendo alterum iusto majorem, alterum iusto mi-
 norem: Sit igitur proxime $\sqrt[n]{A+B \times p} = s \pm \text{fractione}$, $\sqrt[n]{A-B \times p}$
 $= t \mp \text{fractione}$: Hinc $x^2 + y^2 = \frac{s+t}{2}$, & $x^2 = \frac{2r+s+t}{4}$, $y^2 =$
 $\frac{s+t-2r}{4}$, unde radix quæsitæ erit $\frac{\sqrt{2r+s+t} + \sqrt{s+t-2r}}{2 \sqrt[n]{p}}$.

Hæc regula, ut dicit Eulerus, semper radices quæsitæ, si modo ullæ
 sint, detegit. Si vero data quantitas $A+B$ multiplicetur in 2^n , tum
 quantitatis resultantis per Schootenii regulam semper extrahi possit ra-
 dix quæsitæ, si modo ea radix per Euleri regulam inveniri possit, e. g.

Sit data quantitas $2 + \sqrt{5}$, cujus radix cubica erit $\frac{1+\sqrt{5}}{2}$, quæ

per Schootenii methodum haud inveniri possit; ducatur $2 + \sqrt{5}$ in
 2^3 , & radix cubica quantitatis $16 + 8\sqrt{5}$ per Schootenii methodum
 extrahi possit.

Hæ autem regulæ fallunt, cum una quantitas A vel B sit impossi-
 bilis: hoc in casu pone $A-B \times p = r^n$, & sit radix quæsitæ $\frac{x+y}{2 \sqrt[n]{p}}$:

assumatur æquatio $z^n - n r^2 z^{n-2} + n \cdot \frac{n-3}{2} r^4 z^{n-4} - n \times \frac{n-4}{2} \times$
 $\frac{n-5}{3} r^6 z^{n-6} + \&c. = 2p \times \sqrt[n]{A+B}$, ex quâ valor ipsius z innotescet,

& erit $x^2 + y^2 = \frac{1}{2} z$ & $x^2 - y^2 = r$, unde $x = \frac{\sqrt{z+2r}}{2}$ & $y =$
 $\frac{\sqrt{z-2r}}{2}$

primam quotientem 1, erit $\frac{1}{1}$ minor quam fractio quæsitæ; sed quoniam denominator per primum reliquum divisus dedit secundam quotientem 6, & consequenter reliquum erit minus quam sexta pars denominatoris, & fractio $(1 + \frac{1}{6} = \frac{7}{6})$ major quam vera: sed quoniam primum reliquum per secundum divisum dedit quotientem 2, ergo quoniam duæ proxime præcedentes fuerunt $\frac{7}{6}$ & $\frac{1}{1}$, erit $\frac{7 \times 2 + 1}{6 \times 2 + 1} = \frac{15}{13}$ proxime minor quam vera fractio; & quoniam 7 est subse-

quens quotiens $\frac{15 \times 7 + 7}{13 \times 7 + 6} = \frac{112}{97}$ erit fractio proxime major quam vera; & quoniam divisores respective fuerunt 1, 6, 2, 7, 4, 2, 1, 7, fractiones erunt

$$\begin{array}{l} \frac{1}{1}, \quad \frac{7 \times 2 + 1}{6 \times 2 + 1} \left(\frac{15}{13} \right), \quad \frac{112 \times 4 + 15}{97 \times 4 + 13} \left(\frac{463}{401} \right) \\ \frac{6 \times 1 + 1}{6} \left(\frac{7}{6} \right), \quad \frac{15 \times 7 + 7}{13 \times 7 + 6} \left(\frac{112}{97} \right), \quad \frac{463 \times 2 + 112}{401 \times 2 + 97} \left(\frac{1038}{899} \right) \\ \frac{1038 \times 1 + 463}{899 \times 1 + 401} \left(\frac{1501}{1300} \right), \quad \frac{1501 \times 7 + 1038}{1300 \times 7 + 899} \left(\frac{11545}{9999} \right). \end{array}$$

Superior columna fractiones verâ minores denotent, inferior vero verâ majores; ni ultima, quæ est vera fractio: Inter has respectivas facile esset alias interponere: e. g. inter fractiones $\frac{c}{d}$ & $\frac{na+c}{nb+d}$ inter-

ponantur fractiones $\frac{a+c}{b+d}, \frac{2a+c}{2b+d}, \frac{3a+c}{3b+d}, \frac{4a+c}{4b+d}, \&c. : \frac{n-1a+c}{n-1b+d}.$

Hanc methodum approximationum primus invenit Wallisius.

Cor. 1. Sint $a, b, c, d, e, f, g, \&c.$ prædictæ successivæ quotientes, & respectivæ fractiones erunt $\frac{a}{1}, \frac{ab+1}{b}, \frac{abc+a+c}{bc+1}, \frac{abcd+ab+cd+ad+1}{bc d+b+d},$
 $abcde$

$\frac{abcde + abc + abe + ade + cde + a + c + e}{bcde + bc + de + be + 1}$; & sic deinceps. Numerator & denominator hujusce seriei semper eandem observant legem; alter vero unam literam semper obtinet, qui non invenitur in altero, ea litera est prima quotiens a : Hæc autem litera invenitur in numeratore vel denominatore, prout prima quotiens sit quotiens e divisione numeratoris per denominatorem, vel denominatoris per numeratorem.

In hac serie primus terminus erit $abcdefg$ &c. & omnes reliqui ex hoc cognosci possint; 1^{mo} auferendo continuo duas successivas quotientes, e. g. a & b sunt successivæ quotientes, auferantur e primo termino a & b , & restabit alter terminus quæsitus $cdefg$ &c. b & c sunt successivæ quotientes, auferantur hæ quotientes e primo termino, & restabit $defg$ &c. tertius terminus quæsitus: c & d sunt successivæ quotientes, auferantur e primo termino c & d , & restabit quartus terminus quæsitus efg &c. & sic deinceps: deinde auferantur e primo termino duæ successivæ quotientes, & duæ aliæ successivæ quotientes & restabunt alii termini; deinde auferantur duæ successivæ quotientes ter, quater, &c. usque donec vel solummodo una quotiens vel nulla (in quo casu terminus erit 1) restat: & aggregatum ex omnibus terminis, qui e prædictâ methodo formari possint, erit series quæsitæ.

Cor. 2. Sint duæ fractiones, quarum una habet a figuras in denominatore, b vero in numeratore; altera vero n in denominatore, m vero in numeratore; dividantur respectivi numeratores per suos denominatores, & sit $n + m$ major quam $a + b$, & haud possunt resultare plures quam $n + m$ figuras in quotiente easdem, ni utriusque fractionis idem sit valor: Hinc fractio, quæ habet $n + m + 2$ figuras in quotiente e numeratore per denominatorem diviso resultante easdem, quas habet data series, propior vel æquidistans erit valori seriei quam ulla alia fractio, quæ solummodo habet $n + m$ figuras in ejus numeratore & denominatore conjunctim contentas.

Methodo haud multum dissimili inveniri possint, utrum duæ vel

plures rationales fractiones sint, quæ in datas quantitates in infinitum progredientes ductæ, summam rectangulorum æqualem rationali fractioni producant; & exinde investigare licet, utrum data quantitas in infinitum progrediens exprimi possit per radices fractionum, necne: sed hæc ad summationem serierum pertinent.

THEOR. XI.

Sit fractio $\frac{a}{b} + \frac{\beta}{c}$
 Reducatur hæc $\frac{c + \delta}{d + \epsilon}$
 Fractio in vulgarem $\frac{e + \zeta}{f + \eta}$, &c.
 Fractionem, & resultant

respectivæ fractiones $\frac{a}{a + \beta}$ major quam vera, $\frac{a}{a + \beta} = \frac{b}{b + \beta} \frac{a}{a + \beta}$ minor
 quam vera, & sic alternatim fractiones resultantes majores & minores erunt quam vera: tertia vero invenitur $\frac{a}{a + \beta} = \alpha \times \frac{b}{b + \gamma} \frac{c}{c + \delta}$

$\frac{b c + \gamma}{a b c + a \gamma + c \beta}$, quarta vero $\alpha \times \frac{b c d + b \times \delta + d \times \gamma}{a b c d + a b \times \delta + a d \times \gamma + d c \beta + \delta \beta}$
 quinta vero $\alpha \times \frac{b c d e + b c \times \epsilon + b e \times \delta + \gamma \epsilon}{a b c d e + a b c \times \epsilon + a b e \times \delta + a d e \times \gamma + c d e \times \beta + a \times \gamma \epsilon + e \times \beta \delta + c \beta \epsilon}$
 &c. Pro $\alpha, \beta, \gamma, \delta, \epsilon$, &c. scribatur respective unitas, & eadem erit series ac ea in coroll. i. problem. præceden. tradita. Lex autem, quam observant literæ $\alpha, \beta, \gamma, \delta, \epsilon$, &c. sic enunciari potest. Scribantur termini duarum serierum $a, b, c, d, e, f, \dots l, m$, &c.

$\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \dots \lambda, \mu$, &c. infra se invicem respective; & duæ successivæ literæ prædictæ seriei (ut constat e præcedente problem.) semper simul evanescunt; i. e. vel a & b , vel c

&

& d , vel d & e , vel e & f , &c. pro singulis duabus successivis literis destructis (l & m) prædictæ seriei scribatur unica litera (μ) inferioris ordinis infra posteriorem literam (m) destructam: e. g. in quinta fractione primus terminus numeratoris est $b c d e$, secundus vero $b c \times e$; in hoc termino destruuntur duæ successivæ literæ $d e$; ergo in hoc termino pro literis destructis d, e , scribenda est litera e , quæ infra posteriorem literam (e) superioris seriei (a, b, c, d , &c.) destructam in posteriori serie ($\alpha, \beta, \gamma, \delta$, &c.) ponitur.

Hæ fractiones etiam ex hâc methodo inveniri possint: Scribantur respective infra se

$$\begin{array}{ccccccc} a, & b, & c, & d, & e, & & \&c. \\ \frac{a}{a}, & \frac{b a}{a b + \beta}, & \frac{b c a + \gamma a}{a b c + a \gamma + c \beta}, & \frac{b c d a + b \delta a + d \gamma a}{a b c d + a b \delta + a d \gamma + c d \beta + \beta \delta}, & & & \&c. \\ \alpha, & \beta, & \gamma, & \delta, & \epsilon, & \zeta, & \&c. \end{array}$$

& cujusque fractionis numerator per literam supra scriptam multiplicatus una cum numeratore præcedentis fractionis per suam infra scriptam literam multiplicato præbet numeratorem sequentis fractionis: atque eodem modo cujusque fractionis denominator per literam suam supra positam multiplicatus una cum denominatore præcedentis fractionis per literam suam infra scriptam multiplicato præbet denominatorem fractionis sequentis.

Cor. 1. Inveniantur prædictarum fractionum differentiæ; i. e. subtrahatur quæque fractio e præcedente, & resultabunt respectivæ frac-

tiones $\frac{a}{a}, -\frac{\beta a}{a \times a b + \beta}, \frac{\alpha \beta \gamma}{a b + \beta \times a b c + a \gamma + c \beta},$ &c. e quibus facile constabit lex, quam observant series, quæ has differentias exprimant.

Cor. 2. Sit fractio $\frac{A}{B}$ in continuam fractionem transmutanda: dividatur A per B , fitque quotus a & residuum C ; per hoc residuum (C) dividatur præcedens divisor, & fit quotus b & residuum D ; i. e. fit operatio

$$B | A | a$$

$B | A | a$
 $C | B | b$
 $D | C | c$
 $E | D | d$
 &c.

Hanc methodus primum invenit celeberrimus Vice comes Bröunker: Plurima de his rebus & elegantissima adjecit vir clarissimus Euler, cum vero in infinitas series excurrant, ad tractatum de hisce seriebus sunt referenda.

PROB. XLI.

Datâ æquatione $ax + by + cz + dv + \&c. + f = 0$, in quâ incognitæ quantitates $x, y, z, v, \&c.$ unam solummodo habent dimensionem, invenire integros earum correspondentes valores.

Sit coefficientis (a) termini (ax) minima, quæ invenitur in datâ æquatione, dividantur reliqui termini $by, cz, dv, \&c.$ per hanc coefficientem a ; & sit residuum $\dot{b}y$ e divisione termini (by) per coefficientem a ; residuum $\dot{c}z$ e divisione termini (cz) per coefficientem a ; & sic de reliquis; i. e. sit aggregatum residuorum $\dot{b}y + \dot{c}z + \dot{d}v + \&c. + \dot{f}$: assumatur nova æquatio $au + \dot{b}y + \dot{c}z + \dot{d}v + \&c. + \dot{f} = 0$; sit $\dot{b}$ minima coefficientis, quæ invenitur in novâ æquatione; dividantur reliqui $au, \dot{c}z, \dot{d}v, \&c.$ per hanc coefficientem $\dot{b}$, & sint residua e divisione terminorum $au, \dot{c}z, \dot{d}v, \&c.$ per hanc coefficientem respective $\ddot{a}u, \ddot{c}z, \ddot{d}v, \&c.$ & eorum aggregatum $\ddot{a}u + \ddot{c}z + \ddot{d}v + \&c. + \ddot{f}$ supponatur nova æquatio $\dot{b}\phi + \ddot{a}u + \ddot{c}z + \ddot{d}v + \&c. + \ddot{f} = 0$: & ita repetatur hæc operatio, usque donec residuum nihilo sit æquale: deinde pro respectivis ultimæ æquationis quantitatibus assumantur quicunque integri numeri, & ope æquationum hinc deductarum invenientur incognitarum quantitatum ($x, y, z, v,$) correspondentes integri valores.

Hæc

Hæc resolutio melius intelligi possit per exempla.

Ex. 1. Sit $16y + 20 = 7x$, invenire integros quantitatum (x & y) correspondentes valores.

dividatur $16y + 20$ per 7 minorem coefficientem, & operatio erit

$$\begin{array}{r} 7 \overline{) 16y + 20} \quad 2y + 2 \\ \underline{14y + 14} \\ 2y + 6 \end{array}$$

residuum erit $2y + 6$: Supponatur $2y + 6 = 7v$: dividatur hæc nova æquatio per minorem coefficientem 2, & operatio est $2 \overline{) 7v - 6}$

$$\begin{array}{r} 2 \overline{) 7v - 6} \\ \underline{6v - 6} \\ v \end{array}$$

assumatur nova æquatio $v = 2w$; & e secundâ æquatione $2y - 6 = 7v = 14w$, & exinde $y = 7w - 3$; e vero datâ æquatione $16y + 20 = 112w - 28 = 7x$ sequitur $x = 16w - 4$; unde correspondentes valores quantitatum (x & y) erunt $16w - 4$ & $7w - 3$ respective.

Ex. 2. Sit $8x + 12y + 20 = 15v$: dividatur per minimam coefficientem 8, & operatio erit $8 \overline{) 12y + 20 - 15v} \quad y + 2 - v$

$$\begin{array}{r} 8 \overline{) 12y + 20 - 15v} \\ \underline{8y + 16 - 8v} \\ 4y + 4 - 7v \end{array}$$

residuum $4y + 4 - 7v$ assumatur $8z + 4y + 4 = 7v$, quæ est secunda æquatio: dividatur per 4 minimam coefficientem, & operatio erit $4 \overline{) 8z + 4 - 7v} \quad 2z + 1 - v$

$$\begin{array}{r} 4 \overline{) 8z + 4 - 7v} \\ \underline{8z + 4 - 4v} \\ 3v \end{array}$$

residuum $= -3v$: assumatur $4w = 3v$, quæ est tertia æquatio: dividatur per 3 minorem coefficientem, & resultat $3 \overline{) 4w} \quad w$

$$\begin{array}{r} 3 \overline{) 4w} \\ \underline{3w} \\ w \end{array}$$

assumatur $w = 3u$ quarta æquatio; & exinde e tertiâ æquatione $4w = 12u = 3v$, & $v = 4u$; e secundâ vero $8z + 4y + 4 = 7v = 28u$, & exinde $y = 7u - 1 - 2z$; e primâ vel datâ vero æquatione $8x + 12y + 20 = 8x + 84u - 24z - 12 + 20 = 15v = 60u$, unde $x = 3z - 1 - 3u$, & correspondens valor incognitæ quantitatis $y = 7u - 2z - 1$, & quantitatis $v = 4u$; & pro z & u assumi possint quicunque integri numeri.

Cor.

Cor. 1. Sit (m) numerus incognitarum quantitatum in data æquatione & $(m-1)$ quantitates plerumque resultant, pro quibus assumi possint quicunque integri numeri.

Cor. 2. Sit $\alpha x + \beta y + \gamma z + \delta v + \&c. = A$, & si datae coefficientes $\alpha, \beta, \gamma, \delta, \&c.$ communem habeant divisorem, qui haud etiam dividat numerum A , tum nullos integros valores incognitarum quantitatum $x, y, z, v, \&c.$ correspondentes admittet data æquatio.

Cor. 3. Datis limitibus, inter quos consistant quæcunque incognitæ quantitates: substituantur hi limites pro suis quantitatibus in datâ & resultantibus æquationibus, & exinde constabunt omnes respectivi limites, qui exinde sequuntur: e. g. si requiratur, ut quæcunque quantitas semper sit affirmativus, & integer numerus: minimus autem integer numerus est 1, ergo scribatur 1 pro eâ quantitate & in datâ & resultante æquatione, & exinde constabunt limites, quos præbet prædictus limes 1. Hinc etiam constabit numerus valorum inter quoscunque duos limites contentorum.

Cor. 4. Sit æquatio $a x + b y = P$, & coefficientium a & b sit π maximus divisor: & sint p & q correspondentes valores incognitarum quantitatum x & y : & correspondentes valores incognitarum quantitatum x & y erunt respectivè

$$p, p + \frac{b}{\pi}, p + \frac{2b}{\pi}, p + \frac{3b}{\pi}, p + \frac{4b}{\pi}, \&c. \quad \text{Dato igitur correspondente valore e singulis incognitis quantitatibus } x \text{ & } y, \text{ facile deduci possint omnes ejusdem generis valores: unus autem valor vel e priori methodo inveniri possit; vel si modo } a \text{ & } b \text{ nullum accipiant communem divisorem, & maxima coefficiens (a) per minimam } b \text{ dividatur & divisor per residuum, & sic deinceps; usque donec ad aliquod residuum perveniatur, quod dividat numerum } P: \text{ & sint } \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \&c. \text{ respectivæ quotientes ex hujusmodi divisione, & scribantur}$$

$\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \&c.$ & sint $A, B, C, D, E, \&c.$ & scribantur

$$+1, -\alpha, \beta A + 1, -\gamma B - A, \delta C + B, -\epsilon D - C, \zeta E + D, \&c.$$

ubi

ubi A, B, C, D, E , &c. præcedentes terminos respective denotent: multiplicetur ultimus terminus inferioris columnæ per P , & productum erit valor quantitatis y quæsitus: si vero minor valor quæsitus sit, dividatur prædictum productum per quantitatem a & residuum erit æqualis vel minor valor quantitatis y .

Ex. Sit $256x - 87y = 50$ & dividatur 256 per 87, & divisor 87 per residuum & sic deinceps; & respectivæ quotientes inveniuntur 2, 1, 16, 2, 2, & exinde 1, 2, $2 \times 1 + 1 = 3$, $16 \times 3 + 2 = 50$, $50 \times 2 + 3 = 103$; ducatur 103 in $P = 50$, & resultat valor incognitæ quantitatis $y = -50 \times 103 = -5150$ & ejus correspondens valor quantitatis $x = -1750$: minimi vero affirmativi integri valores inveniuntur 30 & 10 respective.

Cor. 1. Quantitates A, B, C, D, E , &c. erunt respective $a, a\beta + 1, a\beta\gamma + a + \gamma, a\beta\gamma\delta + a\beta + a\delta + \gamma\delta + 1, a\beta\gamma\delta\epsilon + a\beta\gamma + a\beta\epsilon + a\delta\epsilon + \gamma\delta\epsilon + a + \gamma + \epsilon$, &c. cujus lex eadem erit ac ea quæ in problem. 40. cor. 4. docetur.

Cor. 2. Sint c, d, e, f, g , &c. respectiva residua e divisione coefficientis a per b , & divisoris b per primum residuum, & sic deinceps; usque ad residuum 1, tum unus valor quantitatis y erit $= aP \times$

$\frac{1}{ab} - \frac{1}{bc} + \frac{1}{cd} - \frac{1}{de} + \frac{1}{ef} - \&c.$ eousque hâc serie continuatâ, donec quantitas P per factorem aliquem denominatoris dividi possit.

PROB. XLII.

Sint duæ æquationes vel tres vel plures (m) incognitas quantitates habentes, quæ unam solummodo habent dimensionem, eas in unam transformare, ut exterminetur una incognita quantitas (x), & si modo incognitæ quantitates resultantis æquationis sint integri numeri, exterminata quantitas (x) etiam erit integer numerus.

1. Sint duæ æquationes $ax + \beta y + \&c. = A$
 $\pi x + \rho y + \&c. = B$
 assumatur æquatio $am - \pi n = 1$, & per problema præcedens inveniantur

A a

niantur

niantur integri valores quantitatum m & n , qui sint r & s respective; ducatur prior data æquatio in r , posterior vero in s , & resultant $r\alpha x + r\beta y + \&c. = rA$, & $s\pi x + s\rho y + \&c. = sB$, quarum differentia erit $x + r\beta - s\rho y + \&c. = rA - sB$ & exinde $x = rA - sB + s\rho - r\beta \times y$; quâ quantitate pro suo valore (x) in unâ datâ æquatione substitutâ, & fit problema.

Ex. Sint duæ æquationes $7x + 6y + 4z = 12$ & $5x + 10y + 12z = 20$, eas transformare in unam, ita ut exterminetur incognita quantitas (x): assumatur æquatio $7m - 5n = 1$; & 3 & 4 erunt integri correspondentes valores incognitarum quantitatum m & n ; ducatur prior data æquatio in 3, posterior vero in 4, & resultant duæ æquationes $21x + 18y + 12z = 36$, & $20x + 40y + 48z = 80$, quarum differentia erit $x - 22y - 36z = -44$ & exinde $x = 22y + 36z - 44$, quo valore quantitatis (x) in primâ æquatione substituto, resultat $160y + 256z - 308 = 12$, i. e. $20y + 32z = 40$.

2. Si quantitates α & π communem habeant divisorem, qui haud sit etiam divisor unius vel alterius æquationis, e. g. $\alpha x + \beta y + \&c. = A$, tum e præcedenti methodo minime resolvi possit problema.

In hoc casu inveniantur ex alterâ æquatione valores e singulâ incognitâ quantitate per præcedens problema; quibus valoribus pro suis quantitibus in alterâ æquatione substitutis, resultat æquatio quæsitâ; e quâ inveniri possunt correspondentes valores e singulis incognitis quantitibus.

Ex. Sint duæ æquationes

$$\begin{aligned} 6x + 20y - 8z + 9v &= 33 \\ &\& 9x + 14y + 12z - 21v = 99 \end{aligned}$$

eas in unam transformare, ita ut exterminetur quantitas x .

E priori æquatione inveniantur correspondentes valores e quantitatibus x, y, z, v ; hi autem erunt $v = 1 - 2w$; $y = 3w - 3u + z$, $x = 10u - 7w - 2z + 4$; ubi u, z, w quoscunque integros numeros assumptos respective denotent, quibus valoribus pro suis respectivis quantitibus in secundâ æquatione substitutis; resultat 21

$w +$

$w + 48u + 8z = 84$; & transformentur duæ æquationes in unam, ita ut exterminetur incognita quantitas (x).

Cor. E resultanti æquatione invenitur $w = -4 + 8a$; & $z = 21 - 21a - 6u$, ubi a quemlibet numerum denotet, & exinde $v = 9 - 16a$, $y = 3a - 9u + 9$, $x = 22u - 14a - 10$.

P R O B. XLIII.

1. Invenire æquationem duas incognitas quantitates (x & y) habentem, & cujus correspondentes valores incognitarum quantitatuum (x & y) sint respective $x = A z^n + B z^{n-1} + C z^{n-2} + \&c.$ & $y = a z^m + b z^{m-1} + c z^{m-2} + \&c.$ ubi n æqualis vel major est quam a .

Assumatur æquatio $P y^m + p + q x y^{m-1} + r + s x + t x^2 y^{m-2} + \&c. = 0$, ita ut $(m \cdot \frac{m+3}{2})$ æqualis vel major sit quam $(n m + 1)$:

In assumptâ æquatione pro incognitis quantitatibus (x & y) substitu-
antur earum respectivi valores $A z^n + B z^{n-1} + C z^{n-2} + \&c.$ & $a z^m + b z^{m-1} + c z^{m-2} + \&c.$ æquationis ($n \times m$) dimensionum re-
sultantis omnium terminorum singulæ coefficientes nihilo æquales fi-

ant: & fit $(m \times \frac{m+3}{2})$ major quam $n m + 1$, tum infinitæ; fit æ-
qualis, tum una solummodo deduci potest hujusce generis æquatio,
in quâ incognitarum quantitatuum (x & y) correspondentes valores sint
datæ quantitates.

Cor. Assumi possint infinitæ diversæ æquationes; quæ hoc proble-
ma resolvant: tot vel plures incognitas coefficientes solummodo ha-
bere debet assumpta æquatio, quot dimensiones sint æquationis resul-
tantis e substitutione correspondentium valorum incognitarum quan-
tatum (x & y) in assumptâ æquatione pro quantitatibus ipsis, uni-
tate auctæ.

2. Eodem plane modo inveniri possint æquationes duas incognitas
quantitates (x & y) habentes, & quarum correspondentes valores in-

cognitarum quantitatuum $x = \frac{az^n + bz^{n-1} + cz^{n-2} + \&c.}{az^n + bz^{n-1} + cz^{n-2} + \&c.}$ & $y = \frac{Az^n + Bz^{n-1} + Cz^{n-2} + \&c.}{Az^n + Bz^{n-1} + Cz^{n-2} + \&c.}$ Et sic de infinitis consimilibus.

Cor. 1. Hinc facile inveniri possint infinitæ æquationes, duas incognitas quantitates (x & y) habentes, & quarum (x & y) correspondentes valores generaliter per quantitates quarumlibet formularum, in quibus solummodo una continetur incognita quantitas (z), exprimi possint.

Cor. 2. Sint $(n - 1)$ incognitæ quantitates in datis correspondentibus valoribus e singulis incognitis quantitatibus ($x, y, \&c.$) contentæ: assumatur æquatio (n) incognitas quantitates ($x, y, \&c.$) involvens; ita ut, si modo substituantur in assumptâ æquatione pro singulis incognitis quantitatibus, earum dati correspondentes valores, evanescant omnes resultantis æquationis termini: tales autem æquationes plerumque assumuntur, si modo numerus incognitarum coefficientium assumptæ æquationis æqualis vel major sit quam numerus terminorum resultantis æquationis.

P R O B. XLIV.

Datâ æquatione duas incognitas quantitates (x & y) habente, invenire rationales correspondentes valores incognitarum quantitatuum (x & y), si modo hi correspondentes valores per algebraicam rationalem functionem assumptæ incognitæ quantitatibus (z) generaliter exprimi possint.

Sit æquatio $P y^n + p + q x y^{n-1} + r + s x + t x^2 y^{n-2} + \&c. = 0$, pro incognitis quantitatibus (x & y) substituantur respective $x = \frac{Az^n + Bz^{n-1} + Cz^{n-2} + \&c.}{Pz^n + Qz^{n-1} + Rz^{n-2} + \&c.}$ & $y = \frac{az^n + bz^{n-1} + cz^{n-2} + \&c.}{Pz^n + Qz^{n-1} + Rz^{n-2} + \&c.}$ si solummodo rationales incognitarum

rum quantitatum (x & y) valores quæſiti fuerunt, in datâ æquatione pro ſuis valoribus; & æquationis resultantis fiant ſinguli termini nihilo reſpectively æquales, & ex æquationibus resultantibus inveniantur valores incognitarum coefficientium quæſiti.

Ex. Sit æquatio $4y^3 - 4xy + x^3 + 7y - 4x - 4 = 0$; ſcribantur pro incognitis quantitibus (x & y) reſpectively $y = z^3 + Bz + C$, & $x = az^3 + bz + c$, & reſultat æquatio $4 - 4a + a^3 \times z^4 + 8B - 4aB - 4b + 2ab \times z^3 + 8C - 4c - 4aC + 2ac + 4B^2 - 4Bb + b^2 + 7 - 4a \times z^2 + 8BC - 4bC - 4Bc + 2cb + 7B - 4b \times z + 4C^2 - 4Cc + c^2 + 7C - 4c - 4 = 0$. Fiant coefficientes resultantis æquationis nihilo reſpectively æquales, & invenientur coefficientium (a, b, c, B, C) valores reſpectively (2, 11, 1, 6, 1).

In hoc caſu cum evaneſcat primus terminus z^4 , evaneſcet etiam z^3 , & exinde plures reſultant incognitæ quantitates quam æquationes; & infinitas diverſas huiusmodi reſolutiones admittet data æquatio.

Et ſic ratiocinari licet de rationalibus valoribus incognitarum quantitatum inveniendis.

Cor. Sit data æquatio $p y^n + a + b x y^{n-1} + c + d x + e x^2 y^{n-2} + k x^3 + \&c. y^{n-3} + \&c. = 0$; ſi termini $p y^n + b x y^{n-1} + e x^2 y^{n-2} + \&c.$ in quibus maximæ inveniuntur dimenſiones, nullum ſimplicem rationalem admittant diviſorem, tum minime correfpondentes incognitarum quantitatum (x & y) valores generaliter exprimi poſſint per algebraicam integram vel rationalem functionem aſſumptæ quantitatis (z).

Si admittant ſimplicem rationalem diviſorem prædicti termini, qui ſit $y - \alpha x$; & aſſumatur $x = az^n + bz^{n-1} + \&c.$ tum erit $y = a \alpha z^n + \&c.$ Si vero pro x & y aſſumantur reſpectively $\frac{ax^r + bx^{r-1} + \&c.}{px^s + qx^{s-1} + \&c.}$

& $\frac{Ax^r + Bx^{r-1} + \&c.}{Px^s + Qx^{s-1} + \&c.}$ tum erit $\frac{A}{P} = \frac{a}{p} \times \alpha$.

Cum vero unus ſolummodo inveniatur terminus in datâ æquatione, in quo maximæ inveniuntur dimenſiones, tum e methodo convergentes.

tes series inveniendi, deduci possit relatio inter terminos quantitatis (x), in quibus maximæ inveniuntur ejus dimensiones, e. g. sit $x = ax^n + bx^{n-1} + \&c.$ & $y = Az^m + Bz^{m-1} + \&c.$ & e methodo convergentes series inveniendi constabit relatio inter A & a , n & m .

Et sic de relatione inter terminos, in quibus maximæ inveniuntur dimensiones quantitatis (x), cum rationales solummodo requirantur valores incognitarum quantitatum (x & y).

Eadem etiam applicari possint ad (m) æquationes ($m+1$ vel plures) incognitas quantitates habentes.

PROB. XLV.

Datis duabus æquationibus tres vel plures incognitas quantitates (x , y , z , &c.) habentibus, eas in unam reducere, ita ut exterminetur incognita quantitas (x), & si incognitæ quantitates in resultanti æquatione sint rationales, exterminatæ etiam erunt rationales quantitates.

Reducantur datæ æquationes per methodum primam in prob. 27 traditam, & si tandem perventum sit ad æquationem $Ax = B$, ubi A & B respective denotent quascunque rationales functiones e singulis incognitis quantitatibus (y , z , v , &c.) præter exterminandam (x); substituatur $\frac{B}{A}$ pro x in alterâ datâ æquatione, & reducentur duæ æquationes in unam, ita ut exterminetur incognita quantitas (x), & si reliquæ incognitæ quantitates (y , z , v , &c.) sint rationales, exterminata x etiam erit rationalis quantitas.

Si vero æquationes per prædictam methodum reductæ, nunquam perveniant ad formulam $Ax = B$; sed tandem perventum sit ad duas resultantes æquationes $Ax^2 + Bx = C$, & $Ax^2 + Bx = D$; ubi A , B , C , D respective denotent quascunque rationales functiones incognitarum quantitatum (y , v , z , &c.) in datâ æquatione contentarum: in hoc casu omnis valor e singulis incognitis quantitatibus (y , v , z , &c.) habet duos valores incognitæ quantitatis (x) sibi ipsi respondentes: si vero radix quadratica quantitatis $B^2 + 4AC$ vel $B^2 + 4AD$ haud extrahi possit, tum exterminetur alia quantitas (y vel v , vel z , &c.), quæ non habet duos valores omni e singulis incognitis

nitis quantitatibus valori respondentes; si talis in datis æquationibus inveniatur quantitas (v), tum tandem ad æquationem $Av = B$ perveniri liceat; sin haud talis incognita quantitas in datis æquationibus inveniatur; tum nunquam per prædictam methodum reduci possint duæ æquationes in unam, ita ut exterminentur incognita quantitas, & si resultantis æquationis incognitæ quantitates sint rationales, exterminata etiam erit rationalis.

Si vero per præcedentem methodum haud reduci possint $(n+1)$ æquationes in $(n$ vel pauciores) æquationes: inveniantur una vel plures incognitæ quantitates in rationalibus terminis reliquarum, si modo fieri possit; substituantur hic valor vel hi valores inventi pro suis incognitis quantitatibus, & reducuntur $(n+1)$ æquationes in n vel pauciores, &c.

Si vero duæ vel (n) incognitæ quantitates inveniri possint rationalibus terminis e reliquis & novæ incognitæ quantitatis assumptæ vel etiam $(n-1)$ assumptarum incognitarum quantitarum, tum exinde reduci possint datæ æquationes in pauciores, &c.

Cor. Sint $n+1$ datæ æquationes (n) incognitas quantitates habentes, & quæ solummodo in (n) æquationibus unam habent dimensionem: Inveniantur ex (n) æquationibus valores (n) incognitarum quantitarum, quibus in unâ reliquâ æquatione pro suis valoribus respectively substitutis, & reducuntur hæ $n+1$ æquationes in unam, ita ut exterminentur (n) incognitæ quantitates, & si incognitæ quantitates resultantis æquationis sint rationales, exterminatæ etiam erunt rationales.

Visum est hoc in loco quædam adjicere de quibusdam diversis methodis rationales valores incognitarum quantitarum in datâ æquatione contentarum correspondentes detegendi.

1. Cum detur æquatio incognitas quantitates (x, y, z, v , &c.) involvens; & in omni datæ æquationis termino vel inveniuntur n vel $n-1$ dimensiones quarumcunque incognitarum quantitarum (x, y, z , &c.): Substituantur pro his incognitis quantitatibus (y, z , &c.) in datâ æquatione respectively $\alpha x, \beta x$, &c. & si modo omnes dimensiones prædictæ sint n , tum e datâ æquatione exterminabitur incognita quantitas

quantitas (x); si modo dimensiones sint n & $n-1$, tum resultabit æquatio unam dimensionem incognitæ quantitatis x solummodo continens: ex hac autem æquatione inveniri possit valor incognitæ quantitatis (x) terminis rationalibus incognitarum quantitatum ($y, z, v, \&c.$).

Cor. Sæpe usui inservit hæc substitutionis methodus, cum duæ vel plures sint datæ hujusmodi æquationes.

Ex. Sint $zy + az + by = v^2$ & $zy + ez + \frac{be}{a}y = w^2$, pro y, v, w per regulam scribantur respective $\alpha z, \beta z, \gamma z$: & resultant

e prioræ æquatione $z = \frac{a + b\alpha}{\beta^2 - \alpha}$, e posteriori vero $z = \frac{a}{\gamma^2 - \alpha}$;

unde $\frac{a}{\beta^2 - \alpha} = \frac{e}{\gamma^2 - \alpha}$: & exinde cognoscitur quantitas (α) terminis quantitatum β & γ ; & sic de infinitis hujuscemodi propositionibus.

2. Sit æquatio $ax^2 + by^2 + cz^2 + \&c. + A = pv^2 + qw^2 + ru^2 + \&c. + B$: ubi A & B quascunque rationales functiones incognitarum quantitatum, in terminis autem quarum functionum vel una vel nulla continetur dimensio incognitarum quantitatum $x, y, z, v, w, u, \&c.$ sit etiam $a + b \times \beta^2 + c \times \gamma^2 + \&c. = p \times \pi^2 + q \times \rho^2 + r \times \sigma^2 + \&c.$

Substituatur $\beta x + y = y, \gamma x + z = z, \&c. \pi x + v = v, \rho x + w = w, \sigma x + u = u, \&c.$ & scribantur hi valores pro suis respectivis valoribus in datâ æquatione, & resultat æquatio, in quâ una solummodo continetur dimensio incognitæ quantitatis x , unde inveniuntur incognitæ quantitates $x, y, z, v, \&c.$ in functionibus incognitarum $y, z, \&c. v, w, u, \&c.$ & A & B .

Ex. 1. Sit $x^2 + y^2 = z^2$, pro z scribatur $x + z$ & resultat æquatio

$$y^2 - z^2 = 2zx, \text{ unde } x = \frac{y^2 - z^2}{2z} \text{ \& } y = \frac{2zy}{2z} \text{ \& } z = \frac{y^2 + z^2}{2z}.$$

Ex. 2.

Ex. 2. Sit $a^2 + b^2 = x^2 + y^2$: assumatur $x = a + z$ & $y = b - z$, quibus substitutis resultat æquatio $2ax + x^2 - 2by + y^2 = 0$, quæ æquatio ad præcedentem methodum pertinet. Termini enim ultra duas dimensiones non ascendunt.

Aliter ducatur data æquatio in z^2 & resultat $a^2 + b^2 - x^2 = z^2$ (v^2) + $z^2 y^2$ (w^2), scribantur pro v & w respective $ax + v$ & $bz - w$, & resultat æquatio $2axv + v^2 - 2bzw + w^2 = 0$, unde $z = \frac{v^2 + w^2}{2bw - av}$.

Cor. Cum duæ vel plures sint datae æquationes, hæc methodus substitutionis, quæ ita transformat quosdam terminos, ut alios destruant, frequenter usui inservit; i. e. vel in simplici vel in duplicatâ, &c. æqualitate maxime præstat, ut termini, in quibus maximæ inveniuntur dimensiones quarumcunque incognitarum quantitatum, pro primâ substitutione fiant inter se æquales, & exinde sæpe deduci possit resolutio quæsitâ.

3. Sæpe vero præstat, ut inveniatur per vulgares æquationum resolutionum methodos e quâcunque quantitate valor: & valoris resultantis fiat irrationalis pars æqualis rationali quantitati: & sic de singulâ irrationali quantitatis resultantis parte separatim vel conjunctim, & exinde in simplici & duplicatâ æqualitate sæpe acquiri possint solutiones.

Ex. 1. Sit $ax^3 + bx^2 + cx + d^2 = v^2$, invenire v & x in rationalibus quantitatibus: fingatur $v = d + \frac{cx}{2d}$ (ut destruantur duo termini $d^2 + cx$); tum erit $d^2 + cx + \frac{c^2 x^2}{4d^2} = d^2 + cx + bx^2 + ax^3$, unde $b + ax = \frac{c^2}{4d^2}$ & exinde constat rationalis valor quantitatis x .

B b

Ex. 2.

Ex. 2. Sint $a^2 z^2 + bz + c = v^2$ & $a^2 z^2 + dz + e = w^2$: Supponantur $az + \alpha$ & $az + \beta$ respective v & w per præcedentem casum, & resultant duæ æquationes $bz + c = 2a \times az + \alpha^2$ & $dz + e = 2a\beta z + \beta^2$, unde $z = \frac{\alpha^2 - c}{b - 2a\alpha} = \frac{\beta^2 - e}{d - 2a\beta}$, & inveniatur radix incognitæ quantitatis (α), hujus radice fiat irrationalis pars æqualis rationali quantitati assumptæ consimili methodo, quâ assumitur rationalis quantitas in præcedente exemplo; & exinde deducetur valor quæsitus quantitatis β .

4. Datis (n) æquationibus ($n + m$) incognitas quantitates habentibus, quæ omnes esse rationales requiruntur; pro ($m - 1$) incognitis quantitatibus assumantur quicunque rationales valores, qui reddant solutionem eo magis faciliem, & exinde sæpe deduci possint rationales valores ($m + n$) incognitarum quantitatuum.

Ex. 1. Sit $x^4 + y^4 + z^4 = v^2$. Fingatur $v = a - x^2$ unde $x^4 = \frac{a^2 - y^4 - z^4}{2a}$; & ut destruantur z^4 & y^4 & solutio eo magis facilis reddatur, fingatur $a = z^2 + y^2$; & resultat $x^2 = \frac{z^2 y^2}{y^2 + z^2}$: supponatur $y^2 + z^2 = w^2$ & inveniuntur valores ex incognitis quantitatibus y, z, w , & fit exemplum.

Ex. Sint $z^2 + a = v^2 = z + \alpha$, $zy + a = \beta^2$, $y^2 + a = w^2 = y + \gamma$: & resultant $y = \frac{a - \gamma^2}{2\gamma} = -2a$, si modo assumatur $\beta = a$; & exinde constat unus valor.

Ex. 3. Sint $zy + a^2 = w^2$, $zv + a^2 = u^2$, $yv + a^2 = x^2$: Fingatur $u = a + tz$ & $x = vyt + s$ & ut solutio eo magis facilis reddatur, assumatur $w = a + vy$; & resultabit $y = \frac{s^2 - a^2}{2avt^2 + 2at - 2syt}$.

Cor. Ut resolutio (n) æquationum, in quibus continentur $n + m$ incognitæ quantitates (x, y, z , &c.) generalis evadat, plerumque valor e singulis incognitis quantitatibus (x, y, z , &c.) exprimi debet per

per rationalem functionem, quæ (*m*) independentes quantitates ad lubitum assumendas involvat.

5. Invento e quâcunque methodo operandi uno valore e singulis incognitis quantitatibus: Substituatur hic valor assumptâ incognitâ quantitate auctus vel diminutus, &c. pro eâdem incognitâ quantitate in singulis æquationibus; & exorientur novæ æquationes, e quibus præcedenti operandi methodo investigentur novi valores e singulis incognitis quantitatibus, & sic deinceps.

Hic haud abs re sit observare, quod æquationes, quæ videntur easdem involvere incognitas quantitates, sæpe vero tanquam independentes æquationes haberi debent: e. g. Sint $2x^2 = y^3$ & $z + x^2 = w^3$: hic utraque æquatio eandem habet incognitam quantitatē x ; si vero pro $z + x$ scribatur v , exoriantur duæ independentes æquationes $2x^2 = y^3$ & $v^2 = w^3$: & earum resolutio est resolutio duorum independentium problematum.

6. Facile inveniri possint infinitæ æquationes plures incognitas quantitates habentes, ita ut deduci possint rationales vel integrales valores e singulis incognitis quantitatibus.

Assumantur resolutiones, e quibus detegi possint rationales valores e singulis incognitis quantitatibus, & exinde formentur æquationes, quibus assumptæ resolutiones respondent; & quod requiritur, factum est.

Ex. 1. Sint $xy + x + y = v^2$ & $z = 1 + x + y \pm 2v$ resolutiones assumptæ; quarum rationales valores cognoscuntur; tum exinde consequuntur $xz + x + z$, $yz + y + z$; $xy + z$, $xz + y$, $yz + x$; $xy + xz + yz$ & $xy + xz + yz + x + y + z$ esse quadrata.

Ex. 2. Sit $P = 0$, cujus incognitarum quantatum rationales valores assumi possint, tum $\mathcal{Q}^2 + \alpha P + \beta P^2 + \gamma P^3 + \&c.$ erit quadratus, $R^3 + \pi P + \rho P^2 + \sigma P^3 + \&c.$ erit cubus, &c. quæcunque sint quantitates pro $\mathcal{Q}$, α , β , γ , &c. R , π , ρ , σ , &c. assumptæ vel cognitæ vel incognitæ, &c. si modo rationales pro iis assumantur functiones: & exinde formari possint quoruncunque æquationes, quarum correspondentes rationales incognitarum quantatum valores facile deduci possint.

7 Haud nunquam resolvi possint hujusmodi propositiones juxta subsequentem methodum.

Reducantur datæ quantitates, si modo fieri possint, in alias formulas, quarum relationes cognoscuntur, & exinde deduci possit resolutio.

Ex. Sit $x^3 + y^3 = v^3$, tum reduci potest in hanc formulam $x + y \times x^2 - xy + y^2 = v^3$, unde facile constat $x + y$ & $x^2 - xy + y^2$ vel esse primos inter se vel pro communi mensurâ habere numerum 3, unde $x + y$ vel $= u^3$ vel $= 3 u^3$; & sic $x^2 - xy + y^2$ vel $= v^3$ vel $= 3 v^3$.

PROB. XLVI.

Datâ æquatione duas vel plures incognitas quantitates (x & y) habente, invenire correspondentes integros incognitarum quantitatum valores.

Inveniatur proximus valor unius incognitæ quantitatis (x) terminis vero alterius (y) & datarum quantitatum, qui valor auctus vel diminutus per assumptam quantitatem substituatur pro suo valore (x); & æquationis resultantis eâdem methodo inveniatur proximus valor assumptæ quantitatis, & substituatur pro assumpta quantitate ejus proximus valor auctus vel diminutus novâ assumptâ quantitate; & sic iteratâ continuo operatione, usque donec inveniatur verus valor unius incognitæ quantitatis terminis vero alterius, & deinde e præcedentibus substitutionibus constabunt valores incognitarum quantitatum datæ æquationis quæsitum.

Ex. Sit $13 a^2 + 1 = x^2$; hinc proximus valor quantitatis (x) erit $3a$, & supponatur $x = 3a + b$; quo pro (x) in datâ æquatione substituto, resultat $4a^2 + 1 = 6ab + b^2$; assumatur $a = b + c$, & exinde $4b^2 + 8bc + 4c^2 + 1 = 6b^2 + 6bc + b^2$ hoc est $2bc + 4c^2 + 1 = 3b^2$; assumatur etiam $b = c + d$ & exinde $3c^2 + 1 = 4cd + 3d^2$, assumatur $c = d + e$ & resultat $2de + 3e^2 + 1 = 4d^2$; assumatur iterum $d = e + f$, quo substituto pro suo valore, resultat $e^2 + 1 = 6ef + 4f^2$; & supponatur $e = 6f + g$ & resultat $6fg + g^2 + 1 = 4f^2$; capiatur $f = g + h$ & resultat $3g^2 + 1 = 2gh + 4h^2$; supponatur $g = h + k$ quo valore pro g in ultimâ æquatione substituto, resultat $4hk + 3k^2 + 1 = 3h^2$; ex quâ æquatione colligitur

tur unum valorem quantitatis $k = 1$, & quantitatis $b = 2$, unde $g = b + k = 3$, $f = g + b = 5$, $e = 6f + g = 33$, $d = e + f = 38$, $c = d + e = 71$, $b = c + d = 109$, $a = b + c = 180$, $x = 3a + b = 649$.

Et sic de correspondentibus integris valoribus incognitarum quantitatum æquationis plures quam duas dimensiones habentis.

Et eadem methodus in æquationem tres vel plures incognitas quantitates habentem applicari possit.

Hæc methodus paululum mutata fractionales correspondentes valores incognitarum quantitatum inveniet.

Sed in quam plurimis casibus hæc methodus continuis tentaminibus haud multum præstabit; e. g. fit $xy = 399$; nulla enim in hoc casu quantitas pro approximatione assumi possit: & sic in quam plurimis casibus.

PROB. XLVII.

Datis duabus vel pluribus (n) æquationibus plures incognitas quantitates (x, y, &c.) habentibus, invenire integros correspondentes valores incognitarum quantitatum (x, y, &c.).

Inveniatur proximus valor unius vel plurium incognitarum quantitatum (n) terminis vero reliquarum & datarum quantitatum: qui valor auctus vel diminutus per assumptam quantitatem, vel si modo proximi valores plurium incognitarum quantitatum assumantur, substituantur hi valores aucti vel diminuti per assumptas quantitates pro eorum respectivis valoribus in singulis æquationibus; & sic deinceps, usque donec inveniantur correspondentes incognitarum quantitatum valores, vel quidam valor cujuscunque quantitatis fit (1) vel 0, &c.

Ex. Sint duæ æquationes $x^2 = y^2 + 5z^2 - 12$ & $7x^2 = 10y^2 + 28z^2 - 69$, assumatur $x = y + 2z - v$, quâ quantitate pro suo valore in datis æquationibus substitutâ, resultant $y^2 + 4yz + 4z^2 - 2yv - 4zv + v^2 = y^2 + 5z^2 - 12$, & exinde $v^2 - 4zv - 2yv = x^2 - 4yz - 12$, & $7v^2 - 28zv - 14yv + 28yz = 3y^2 - 69$:

In

In hoc casu supponatur e priori æquatione $v = x + w$, & e posteriori $v = y + u$; unde in duabus præcedentibus æquationibus pro v scribatur $x + w$ & pro y scribatur $x + w + u$ & resultant duæ æquationes $2x^2 + 2xw + w^2 = 12 + 2xu - 2wu$, & $10x^2 + 20xw + 10w^2 - 8xu + 20wu = 69 - 3u^2$: assumatur quantitas $w = 0$ & exinde $u = 1$ & $x = 3$; & $v = x + w = 3$ & $y = x + w + u = 4$ & $x = y + 2x - v = 7$.

Cor. Eadem methodus applicari possit in (n) æquationes $(n + 1)$ vel plures incognitas quantitates habentes: sed maxime crescit eundo labor: & sæpe nulla dari possit methodus nisi per continua Tentamina.

Hæc methodus etiam fractionales correspondentes valores incognitarum quantitatum tandem inveniet.

P R O B. XLVIII.

Datis uno, duabus vel pluribus integris valoribus e singulis incognitis quantitatibus in datâ æquatione duas vel plures incognitas quantitates habente, & datâ exinde methodo alterum integrum valorem e singulis incognitis quantitatibus deducendi, infinitos integros valores e singulis incognitis quantitatibus deducere.

Repetatur eadem operatio continuo, quæ ex uno vel pluribus valoribus alios deducat, & perficitur problema.

Ex. 1. Sit æquatio $nx^2 + 1 = y^2$ & inveniantur r & s correspondentes integri valores quantitatum $(x$ & $y)$, & alter integer valor quantitatis (x) inveniat $2rs$; ejus vero correspondens valor quantitatis (y) erit $2nr^2 + 1 = 2s^2 - 1$: & (m) integer valor incognitæ

quantitatis (x) erit $2^m s^m - \frac{m-1}{2} \cdot 2^{m-2} s^{m-2} + \frac{m-3}{2} \cdot 2^{m-4} s^{m-4} - \frac{m-5}{2} \cdot 2^{m-6} s^{m-6} + \frac{m-7}{4} \cdot 2^{m-8} s^{m-8} - \&c.$ in r ducta; & correspondens valor incognitæ quantitatis

$$\begin{aligned} \text{titatis } (y) \text{ erit } & 2^m s^{m+1} - \frac{m+1}{2} 2^{m-2} s^{m-1} + \frac{m+1}{2} \times \frac{m+1}{2} 2^{m-4} s^{m-3} \\ & - \frac{m+1}{2} \times \frac{m-3}{2} \times \frac{m-4}{3} 2^{m-6} s^{m-5} + \frac{m+1}{2} \times \frac{m-4}{2} \times \frac{m-5}{3} \times \frac{m-6}{4} \\ & 2^{m-8} s^{m-7} \&c. - \frac{m+1}{2} \times \frac{m-5}{2} \times \frac{m-6}{3} \times \frac{m-7}{4} \times \frac{m-8}{5} \times 2^{m-10} s^{m-9} \\ & + \&c. \end{aligned}$$

Lex autem e quâ deduci possint incognitæ quantitates (x & y), aliter exprimi possit.

Per seriem successivi valores incognitæ quantitatís (x) erunt $r, 2rs, 4rs^2-r, 8rs^3-4rs, 16rs^4-12rs^2+r, \&c.$ sint E, F, G tres successivi valores incognitæ quantitatís x , & erit $2sF - E = G$; sint P, Q, R tres successivi valores incognitæ quantitatís y , & erit $2sQ - P = R$.

Idem valores quantitatís (x) designari possunt per subsequentem

$$\text{methodum } r \times 2s - 1 \frac{1}{1} \times 2s - 1 \frac{2s-1}{2s} \times 2s - 1 \frac{4s^2-2s-1}{4s^2-1}$$

$\times \&c.$ i. e. sit $r = 2$, & $2s = 6 = 5 \frac{1}{1}$: & valor quantitatís x erit

$2 \times 5 \frac{1}{1} \times 5 \frac{5}{6} \times 5 \frac{29}{35} \times 5 \frac{169}{204} \times \&c.$ ubi fractionis cujusque numerator æquat denominatorem suum dempto denominatore proxime præcedente: denominator vero numeratorem æquat termini proxime præcedentis in impropriam fractionem reducti: Hoc problema primus invenit Vice comes Brounker.

Hæ methodi minime inveniant omnes possibiles valores: e quibusdam datis valoribus multis modis investigari possint alii: sed quoniam ex hypothesi nulla dari possit regula, quæ omnes valores contineat: ergo nulla regula inveniri possit, quæ aliam haud magis generalem admittat: & e nullâ generali ratiocinatione deduci possint tales regulæ, ex eâ enim nulli oriuntur limites.

Ex. 2. Sit $ax^2 + bx + c = y^2$; & r & s correspondentes integros valores incognitarum quantitatium (x & y) denotent: assumatur æquatio

quatio $ax^2 + 1 = v^2$, & inveniantur correspondentes integri valores incognitarum quantitarum x & v , qui sint p & q , tum successivi va-

lores incognitæ quantitatis x erunt r , $qr + ps + \frac{b \times q - 1}{2a}$, $2q^2r + 2pqs - r + \frac{b \times q^2 - 1}{a}$, &c. & sint E , F , G respective tres successi-

vi valores hujus seriei & $2qF - E + b \times \frac{q - 1}{a} = G$: & successivi

valores incognitæ quantitatis (y) invenientur s , $ap r + qs + \frac{bp}{2}$, $2a pqr + 2q^2s - s + bpq$: sint tres successivi termini hujus seriei respective P , Q , R , tum $R = 2qQ - P$.

Æquatio $v^2 + A + Bx \times v + Cx^2 + Dx + E = 0$ facile in prædictam $y^2 = ax^2 + bx + c$, vel etiam in hanc $y^2 = ax^2 + d$ mutari possit, nihil enim aliud exigit nisi ut destruaturs secundus datæ æquationis terminus.

Cor. Coefficiens a nunquam debet esse quadratum, si enim hoc fiat, haud infinitos valores incognitarum quantitarum (x & y) admittet data æquatio.

Consimiles facile adjici possint æquationes: Hic forsan haud indigna sit observatu Euleri regula,

Sint r & s valores correspondentes incognitarum quantitarum x & y in æquatione $ax^2 + 1 = y^2$, tum $\frac{s}{r}$ erit quam proxime $= \sqrt{a}$, eo

autem magis appropinquat ad radicem $\sqrt{a}$, quo majores sint valores r & s : sed hoc magis pertinet ad infinitas series.

De proprietatibus integrorum numerorum.

1. Sit quantitas $a^m b^n c^r d^s$ &c. quicunque numerus, ubi a , b , c , d , &c. respective primi sint reipsa & inter se numeri; tum numerus divisorum, quos admittat prædictus numerus, erit $m+1 \times n+1 \times r+1 \times s+1 \times$ &c.

Cor. 1. Hinc datis primis divisoribus datæ quantitatis & eorum potestatibus, facile deduci possit, quot divisores admittat data quantitas.

Cor.

Cor. 2. Hinc facile inveniri possint infiniti numeri, qui datum numerum divisorum admittant, etiamque minimus numerus qui datum numerum divisorum admittat: assumantur enim $a, b, c, d, \&c.$ minimi numeri inter se diversi & indices $m, n, r, s, \&c.$ ita ut $m+1 \times n+1 \times r+1 \times s+1 \times \&c. = \text{dato numero}, \& a^m b^n c^r d^s \&c.$ erit minimus numerus.

Cor. 3. Ubi numerus divisorum sit impar, ibi ille numerus erit quadratus; si numerus divisorum sit impar, tum $m+1, n+1, r+1, s+1, \&c.$ erunt respective impares numeri, & consequenter $m, n, r, s, \&c.$ pares numeri, & exinde quantitas quadratus.

Cor. 4. Sint $1+a+a^2+a^3+a^4 \dots a^m = A, 1+b+b^2+b^3+\dots b^n = B, 1+c+c^2+\dots c^r = C, 1+d+d^2+d^3+\dots d^s = D, \&c.$ & aggregata ex omnibus divisoribus quantitatis $a^m b^n c^r d^s \&c.$ simul adjunctis, erunt $A \times B \times C \times D \times \&c.$

Cor. 5. Sit quantitas $2^n - 1$ primus numerus, & quantitas $2^{n-1} \times 2^n - 1$ erit perfectus numerus; summa enim omnium ejus divisorum per corollarium præcedens erit $(1+2+4+8 \dots 2^{n-1} = 2^n - 1 = A \& 1+2^n - 1 = B) A \times B = 2^n \times 2^n - 1$: ex hoc facto subtrahatur numerus ipse $2^{n-1} \times 2^n - 1$, & residuum erit $2^{n-1} \times 2^n - 1$: ergo summa omnium divisorum præter numerum ipsum erit æqualis numero ipsi.

Cor. 6. Sint $2^n x$ & $2^n yz$ amiables numeri, ubi x, y, z sint primi numeri: Summa autem divisorum prioris numeri erit $2^{n+1} - 1 + 2^n - 1 \times x = 2^n yz$; posterioris vero $2^{n+1} - 1 + 2^{n+1} - 1 \times y + z + 2^n - 1 \times yz = 2^n x$: e priori æquatione $x = \frac{2^n yz - 2^{n+1} + 1}{2^n - 1}$, &

exinde e posteriori $z = \frac{2^n - 1 \times y + 2^{n+1} - 1}{y - 2^n + 1} = 2^n - 1 + \frac{2^{2n}}{y - 2^n + 1}$; unde y talis primus numerus assumi debet, ut $y - 2^n + 1$ sit divisor numeri 2^{2n} , & z & x sint etiam primi numeri.

Hinc inveniuntur amiables numeri 284 & 220, etiamque 18416
C c
&

& 17296, &c. Eodem fere modo inveniri possint infiniti alii amica-
biles numeri: assumendo pro 2^n & 2^n quoscunque alios numeros, &
pro x, y, z numeros constantes e pluribus primis numeris.

Sint a & b dati numeri, & $x, y, z, \&c.$ $\alpha, \beta, \gamma, \&c.$ primi numeri
quaesiti, & $a \times y \times z \&c.$ & $b \alpha \beta \gamma \&c.$ amica- biles numeri: Sint m &
 n summa divisorum numerorum a & b respective, tum $m \times \overline{x+1} \times$
 $\overline{y+1} \times \overline{z+1} \times \&c. = n \times \overline{\alpha+1} \times \overline{\beta+1} \times \overline{\gamma+1} \times \&c. = a \times y \times z$
 $\&c. + b \alpha \beta \gamma \delta \&c.$

Facile constat, ut ex hoc problemate infinitæ deduci possint confi-
miles propositiones, quales sint methodi inveniendi numeros diversos,
quorum aggregata divisorum sint inter se æqualia: (e. g. si includa-
tur inter divisores numerus ipse, tum 6 & 11; 10 & 17; 14, 15 &
23, &c.) vel hæc aggregata quamcunque inter se habeant relatio-
nem, &c.

2. Sint numeri $a, b, c, d, e, \&c.$ dividantur per α , & sint respecti-
va residua $\pi, \rho, \sigma, \tau, \&c.$ tum residuum numeri $m a + n b + r c +$
 $s d + t e + \&c.$ per α divisi idem erit ac $m \pi + n \rho + r \sigma + s \tau +$
 $\&c.$ per α divisi.

Sint $a, b, c, d, e, \&c.$ respective 1, 10, 100, 1000, &c. divisor $\alpha = 7$,
& inveniantur respectiva residua $\pi, \rho, \sigma, \tau, \&c.$ & hæc methodus erit
ea a Stifelio prolata. Sit vero divisor 9 & fit maxime vulgaris regula
inveniendi, utrum per 9 dividi possit datus numerus, necne.

Sit datus numerus $a + b$ & assumatur $10 c + b = \overline{10 - \pi} \times m$
vel $100 c + b = \overline{100 - \pi} \times m, \&c.$ tum $\frac{1}{10} a - c + m, \&c.$ dividi
potest per π , si modo datus numerus $a + b$ per π dividi possit.

Si pro π scribatur 7, fit methodus clarissimi Kiraast.

3. Sit $\frac{e+f}{x}$ integer numerus, & m divisor quantitatis (e) , & $m - x$

$= n$

$\frac{ne}{m} + f$
 $= n$, tum $\frac{m}{x}$ erit integer numerus; facile constat e substitutione.

Ex. Sit numerus $f e d c b a$ multiplex numeri (x) ubi f, e, d, c, b, a denotant figuras in suis respectivis locis; dicatur $10 - x = n$, duca-
 tur $n \times f$ & addatur e & summæ nf adjiciatur in directum $d c b a$; &

numerus $n f$ erit multiplex numeri (x) quod docuit Fontinelle;
 $e d c b a$

& sic $n f e$, $n f e d$, &c. erunt multiplices numeri (x): 2. Sit
 $d c b a$ $c b a$

vero $100 - x = n$ & numeri $n f$, $n f e$, $n f e d$, &c. erunt
 $e d c b a$ $d c b a$ $c b a$

multiplices numeri (x) si modo numerus $f e d c b a$ fit multiplex nu-
 meri (x). Sit numerus $f e d c b = 41573$ multiplex numeri ($x = 7$)

& $10 - x = 3$; ergo $n f = 3 \times 4 = 13573$, $n f e =$
 $e d c b$ 1573 $d c b$

$3 \times 41 = 12873$, &c. Sit $100 - x = n = 93$ & $n f = 93 \times 4$
 $d c b$ 1573

$= 38773$, $n f e = 93 \times 41 = 38703$, multiplices erunt numeri
 $d c b$ 573

$x = 7$.

3. 2. Sit quilibet numerus $f e d c b a$ dividendus per a : assumo $10 - a$
 vel $100 - a = n$, &c. e. g. sit $100 - a$ tum operatio erit

$100 - a | f e d c b a | f e d c$
 $f e d c - a \times f e d c$

$+ f e d c \times a + b a$ primum residuum, &c. sic continuo
 repetitâ operatione ad veram quotientem continuâ approximatione
 accedere licet.

Sit 5430856 numerus, per numerum 83 dividendus. In hoc casu
 $100 - 83 = 17 = n$, tum operatio erit

prima quotiens 54308.56

$$17 \times \frac{54308}{100} + 56 = 9232.92$$

$$17 \times \frac{9232}{100} + 92 = 1570.36$$

$$17 \times \frac{1570}{100} + 36 = 267.26$$

$$17 \times \frac{267}{100} + 26 = 45.65$$

$$17 \times \frac{45}{100} + 65 = 8.30$$

$$17 \times \frac{8}{100} + 30 = 1.66$$

$$\frac{17 \cdot 1}{100} + 66 = .83$$

&c.

Quotiens 65432

Hæc fuit methodus Joannis Andreæ Castelvetri.

4. Sint x & z quicunque integri numeri, & sint a & b numeri, quorum nullus invenitur communis divisor; tum $ax + bz$ conficere possit ullum numerum, qui superat numerum $a \times b$.

5. Omnis integer numerus vel est triangulus vel e duobus aut tribus triangulis compositus; i. e. constat ex uno, duobus, vel tribus subsequenter numeris 1, 3, 6, 10, &c.

Omnis integer numerus est pentagonus vel e duobus, tribus, quatuor vel quinque pentagonis compositus; i. e. constat ex uno, duobus, tribus, quatuor vel quinque subsequenter numeris, 1, 4, 10, 20, 35, &c. &c.

Omnis integer numerus est quadratus, vel e duobus, tribus vel quatuor quadratis compositus.

Has proprietates invenit, sed non demonstravit Fermat.

Omnis integer numerus vel est cubus, vel e duobus, tribus, 4, 5, 6, 7, 8,

7, 8, vel novem cubis compositus, est etiam quadrato-quadratus vel e duobus, tribus, &c. usque ad novemdecim compositus, & sic deinceps.

Omnis numerus $4n + 2$ divisibilis per duo, non autem per quatuor componitur e duobus vel tribus quadratis.

In genere omnis integer numerus (adhibitis limitibus e quantitatibus datis exortis) compositus est e quodam finito numero quantitarum hujusce generis $ax^m + bx^n + cx^r + \&c.$ vel e quibusdam numeris quantitarum $ax^m + bx^n + cx^r + \&c.$ & $ex^s + fx^t + gx^v + \&c.$ &c.

Cavendum autem est, ne prædictos limites pro nihilo habeamus: forsan prædictæ quantitates haud incipiunt ab unitate; & investigandus est limes a quo generaliter incipiunt; & forsan sint quantitates hujusce generis $3x^4 + 6x^3 + 24$, ubi omnes quantitates divisibiles erunt per 3, & consequenter nullum possint conficere numerum, præter eos, qui dividi possint per numerum 3, &c.

THEOR. XII.

Sit $N = a^2 + mb^2$, & $M = p^2 + mq^2$, ubi a, b, p, q & m integros denotent numeros, tum erit $N \times M = \overline{ap + mbq}^2 + m \times \overline{aq - bp}^2 = \overline{ap - mbq}^2 + m \times \overline{aq + bp}^2$.

THEOR. XIII.

Sit $N = a^2 + rb^2$ & N^{2m+1} & N^{2m+2} componi possint $(m+1)$ diversis modis e quantitatibus hujusce generis $p^2 + rq^2$.

Hæ autem quantitates designari possunt per hanc methodum

$$N^{2m} = \overline{4ra^2b^2 \times a^2 + rb^2}^{2m-2} + \overline{a^2 - rb^2 \times a^2 + rb^2}^{2m-2} = \text{generaliter}$$

$$\frac{\overline{a + \sqrt{-rb}}^{2l} + \overline{a - \sqrt{-rb}}^{2l}}{4} \times a^2 + rb^2 + \frac{\overline{a + \sqrt{-rb}}^{2l} - \overline{a - \sqrt{-rb}}^{2l}}{4} \times a^2 + rb^2$$

$\times a^2 + rb^2$, ubi l sit quicunque integer numerus æqualis vel minor quam m , &

$$N^{2m+1} = a^2 \times \overline{a^2 + rb^2}^{2m} + rb^2 \times \overline{a^2 + rb^2}^{2m} = \text{generaliter}$$

$$\frac{a + \sqrt{-rb}}{4} + \frac{a - \sqrt{-rb}}{4} \times \overline{a^2 + rb^2}^{2m-1} + \frac{a + \sqrt{-rb}}{4} - \frac{a - \sqrt{-rb}}{4} \times \overline{a^2 + rb^2}^{2m-1}$$

Cor. 1. Hinc constat, cum numerus N componatur (n) diversis modis e prædictis quantitatibus, ejus quadratus etiam componitur (n) diversis modis, ejus cubus & quadrato-quadratus ($2n$) diversis modis ex hisce quantitatibus, & ejus $2m+1$ vel $2m+2$ potestas ($m \times n$) diversis modis e quantitatibus $p^2 + r q^2$ componitur: In quibusdam autem casibus duo vel plures valores fiant inter se æquales.

Cor. 2. Sint N, M, R, S , &c. numeri, quorum b sit numerus, e quantitatibus prædictæ formulæ $a^2 + m b^2$ modis (n, r, s, t , &c.) diversis compositi, tum numerus $N^\alpha \times M^\beta \times R^\gamma \times S^\delta \times \&c.$ (modis

$\frac{\alpha+1}{2} \times \frac{\beta+1}{2} \times \frac{\gamma+1}{2} \times \frac{\delta+1}{2} \times \&c. \times n \times r \times s \times t \times \&c.$ diversis, ubi vel α vel β vel γ vel δ , & sit impar numerus) e quantitatibus prædictis $p^2 + m q^2$ componitur. Si vero ullus factor $\alpha+1$ sit impar, tum pro isto factore scribatur $\alpha+2$, & sic de reliquis.

Hic excipiendi sunt ii casus, in quibus duo vel plures valores fiant inter se æquales.

Cor. 3. Hinc inveniri possint numeri, qui e quantitatibus hujusmodi ($p^2 + m q^2$), quoties (n) quis velit, componuntur. Sit enim $2N = \frac{\alpha+1}{2} \times \frac{\beta+1}{2} \times \frac{\gamma+1}{2} \times \&c.$ si $\alpha, \beta, \gamma, \delta$, &c. sint impares numeri: sin unus $\alpha+2$ sit par, tum pro $\alpha+1$ scribatur $\alpha+2$, &c. & assumantur numeri N, M, R, S , &c. solummodo e duobus quadratis compositi; tum $N^\alpha M^\beta R^\gamma S^\delta \&c.$ erit numerus quæsitus. Et sic generalior reddi possit solutio.

Hinc etiam deduci potest, quoties datus numerus e duobus quadratis

quadratis componitur: Sit $N = a \times b \times c \times d \times \&c.$ & cognoscatur quoties $a, b, c, d, \&c.$ constant e duobus quadratis, ergo inveniri ex hoc theoremate potest, quoties numerus N constat e duobus quadratis.

Cor. 4. Sit numerus $a^2 + m b^2 = A = p \times q \times r \times s \times t \times \&c.$ ubi $p, q, r, s, t, \&c.$ sint primi numeri, & erunt $p = \alpha^2 + m \beta^2, q = \gamma^2 + m \delta^2, \&c.$ ni quidam divisores p & q sint α^2 vel $m \beta^2$, & sic de reliquis: unde e prædictis coroll. sequitur; si A nec sit α^2 , nec $m \beta^2$ & admittat solummodo unicum valorem hujusce generis, tum A erit primus numerus.

Cor. 6. Idem applicari possit in quantitates hujusce formulæ $a^2 + m a b + n b^2$, facile enim reduci possint in quantitates præcedentis formulæ, & sic de pluribus hujuscemodi quantitatibus.

1. Denotante m integrum numerum, omnis numerus primus, qui sit $4m + 1$, quales 5, 13, 17, 29, 37, &c. componitur e duobus quadratis.

Omnis primus numerus, qui sit $3m + 1$, quales 7, 13, 19, 31, &c. componitur e quadrato & triplo alterius quadrati.

Omnis numerus primus, qui vel sit $8m + 1$ vel $8m + 3$, quales 1, 17, 41, 73, &c. vel 3, 11, 19, 43, &c. componitur e quadrato, & duplo alterius quadrati: duplum cujuslibet numeri primi, qui sit $8m - 1$, quales sint $2 \times 7, 2 \times 23, 2 \times 31, 2 \times 47, \&c.$ componitur e tribus quadratis.

Duo primi numeri desinentes aut in 3 aut in 7, & qui sint etiam $4m + 3$, quales 3, 23, 43, 83, &c. & 7, 47, 67, 107, &c. inter se ducantur, & productum componitur e quadrato & quintuplo alterius quadrati; e.g. $3 \times 7 = 21$ componitur e quadrato (1) & quintuplo alterius (5×4).

Hæ sunt propositiones, quas invenit Fermat, ut maxime verisimile est, ex inductione; nullas enim dedit demonstrationes, & ex eâ facile esset confirmiles adjicere.

2. Sit n primus numerus, tum $a^{n-1} - 1$ semper dividi potest per n , ni a per n dividi possit. Si enim $a^{n-1} - 1$ dividi possit per n , tum $a^n - a$; si vero $a^n - a$ tum $a + 1 - a - 1$ dividi potest per n : enim $a + 1 = a^n +$

$a^n + na^{n-1} + n \times \frac{n-1}{2} a^{n-2} \dots na + 1$; cujus seriei singuli termini præter primum & ultimum per n dividi possunt, ergo $a + 1 - a^n - 1$ divisionem per n admittet; sed $a^n - a$ divisionem per n admittet, ergo $a + 1 - a - 1$; & exinde, si vera sit propositio, cum a sit quidam numerus, vera etiam erit quicumque numerus sit a .

Hanc proprietatem primum invenit Dom. Beaufort; demonstrationem vero invenit clariss. Euler, & subsequenter adjecit corollaria e præcedenti proprietate facile deducenda.

1. $\frac{a^{n-1} - b^{n-1}}{n}$ erit integer numerus, cum a & b per n dividi nequeunt.

2. Sint m, n, p, q , &c. numeri primi & inæquales, sitque A minimus communis dividorum eorum unitate minorum, puta ipsorum $m-1, n-1, p-1, q-1$, &c. tum a^A divisa per $mnpq$ &c. vel 0 vel 1 relinquit: nisi a dividi possit per aliquem horum numerorum m, n, p, q , &c.

3. Sit n primus numerus, & omnis potentia, cujus exponentis est $n^{n-1} \times n - 1$, divisa per n^n , vel 0 vel 1 relinquit.

4. Sit $2n + 1$ primus numerus & $n = 4p + 1$, ubi p sit integer numerus, tum $\frac{2^n + 1}{2n + 1}$ erit integer numerus si $n = 4p + 1$ vel $4p + 2$;

sin aliter, $\frac{2^n - 1}{2n + 1}$ erit integer numerus; & $\frac{3^n + 1}{2n + 1}$, si $n = 6p + 2$ vel

$6p + 3$; sin aliter $\frac{3^n - 1}{2n + 1}$ erit integer numerus.

5. Sit p maximus communis divisor quantitatum m & $2n$, tum $a^p - b^p$ divisibilis erit per $2n + 1$, si $a^m - b^m$ divisibilis sit per $2n + 1$ primum numerum.

His casibus Euleri adjungere liceat, $\frac{4^n - 1}{2n + 1}$ semper fore integrum numerum, & in genere idem constat pro omnibus quadratis numeris.

6. Sint

6. Sint residua e divisione quantitatum $a^n, b^n, c^n, d^n, \&c.$ per $2n+1$ respective $\alpha, \beta, \gamma, \delta, \&c.$ In hoc autem casu $\alpha, \beta, \gamma, \delta, \&c.$ erunt 1 vel -1 , & residuum e divisione compositi numeri $(a^n b^n c^n d^n \&c.)$ per $2n+1$ erit $\alpha \times \beta \times \gamma \times \delta \times \&c.$

Sint prædicta residua e divisione a^n & b^n per $2n+1$ respective α & β , cum $mx + \sigma = n$ & $ry + \tau = n$ respective; æquatis autem terminis $mx + \sigma = ry + \tau$, proble. 41 ope inveniri possit casus, in quo residuum e divisione $a^n b^n$ per $2n+1$, erit $\alpha \beta$; e. g. numerus 6 componitur e numeris 2 & 3, cum autem $n = 4p + 2$ & $n = 6q + 2$, residua e divisionibus quantitatum 2^n & 3^n per $2n+1$ respective erunt -1 & -1 ; ergo residuum e divisione quantitatis 6^n per $2n+1$ erit 1×1 , cum $n = 4p + 2 = 6q + 2$; unde fit $q = 2w$, & exinde $6q + 2 = 12w + 2 = n$: ergo residuum erit 1, cum $n = 12w + 2$, &c.

Iisdem positis residuis α & β , & æquationibus $mx + \sigma = n$ & $ry + \tau = n$: ex æquatione $mx + \sigma = ry + \tau$ constat etiam casus, in quo residuum e divisione $a^n b^n$ per $2n+1$ erit $\alpha \pm m\beta$: sed de his nimium dictum est. Et sic de pluribus terminis.

PROB. XLIX.

Invenire omnes divisorum formulas e quibuscunque datis quantitibus.

Ita dividatur omnis data quantitas in alias, quarum divisorum formulæ per notas regulas facile deduci possint: & exinde sequuntur formulæ e datis quantitibus quæsitæ.

Cor. 1. Literæ (x, a, b, c, m, n) integros semper denotent numeros: tum numerus $4x - 1$ nunquam possit esse $= a^2 + b^2$; & primi divisores quantitatis $a^2 + b^2$ vel continentur in formula $4x + 1$, vel fit 2; ni a & b per numerum $4x - 1$ dividi possint respective: omnes primi divisores quantitatis $a^{2^m} + b^{2^m}$ vel continentur in formulâ $2^{m+1}x + 1$ vel fit 2; ni a & b per numerum $4x - 1$ dividi possint respective.

Cor. 2. Sint a & b inter se primi numeri, & $a^2 + nb^2 = A$, ubi n fit primus numerus, tum numerus formularum ejus divisorum $(4nx + a)$ erit $n - 1$, tot enim valores habet quantitas a ; si vero n fit hu-

hujusce formulæ $(4x - 1)$ quantitas; tum hæ $(n - 1)$ formulæ $(4nx + a)$ semper reduci possint in $\frac{n-1}{2}$ numerum formularum hujusce generis $(2nx + a)$. Sit $n = 2$ & erit $4nx + a = 8x + 1$ & $8x + 3$ respective. Divisores 2 & n etiam habet data æquatio.

Sit $n = p \times q \times r \times s \times \&c.$ ubi $p, q, r, s, \&c.$ primos numeros respective denotent, tum numerus divisorum formularum $(4nx + a)$ erit $= \overline{p-1} \times \overline{q-1} \times \overline{r-1} \times \&c.$ fit vero $n = 2 p q r \&c.$ tum numerus prædictus erit $2 \times \overline{p-1} \times \overline{q-1} \times \overline{r-1} \times \&c.$ vel divisores erunt vel 2, vel $p, q, r, \&c.$

Sit $4nx + a$ formula divisoris quantitatis $a^2 + nb^2$, tum $4nx - a$ minime possit esse formula divisoris prædictæ quantitatis.

Omnis formula $(4pqx + a)$, quæ contineat divisores quantitatis $(a^2 + pqb^2)$, etiam erit formula, quæ contineat divisores quantitatis $(pa^2 + qb^2)$.

Sit $4nx + a$ formula, quæ contineat divisores hujusce formulæ $a^2 + nb^2$, tum omnis primus numerus $A = 4nx + a$ erit etiam hujusce formulæ $a^2 + nb^2$.

Sit $a^2 - nb^2 = A$, ubi $n = p \times q \times r \times s \times \&c.$ vel $= 2 p q r s \&c.$ etiamque $p, q, r, s, \&c.$ primos denotent numeros; tum numerus divisorum $(4nx + a)$ erit ac in præcedente casu vel $\overline{p-1} \times \overline{q-1} \times \overline{r-1} \times \&c.$ vel $2 \times \overline{p-1} \times \overline{q-1} \times \overline{r-1} \times \&c.$

Si hoc in casu $4nx + a$ fit formula divisorum, tum $4nx - a$ erit etiam formula ejus divisorum.

Sit $n = 4x + 1$, tum formulæ divisorum exprimi possint per formulam $2nx \pm a$: erunt etiam primi divisores 2, $p, q, r, \&c.$

Sit q primus numerus, & omnes divisores quantitatis $a^2 - 1$ præter $a - 1$, erunt hujusce formulæ $2qx + 1$.

Facile etiam investigari possint omnes diverfi valores quantitatis (a) , quorum unus semper erit 1.

Si n haud fit aggregatum $= a^2 + \frac{1}{2} b \times \overline{b+1}$, tum $8n + 1$ haud fit $8a^2 + 4b^2 + 4b + 1 = 2a^2 + \beta^2$ & non potest esse primus numerus.

Si

Si n haud sit $= \frac{1}{2} b \times \overline{b+1} + a \times \overline{a+1}$, tum $8n+3$ haud sit $= 4b^2 + 4b + 8a^2 + 8a + 3 = 2c^2 + d^2$ & consequenter haud sit primus numerus: & sic deinceps.

Hinc facile inveniri possint multi numeri, qui nunquam possint esse quadrati: Inveniatur quantitatis formula; quæ nunquam possit esse divisor quantitatis $a^2 \pm nb^2$; e formulâ quantitatis subtrahatur, &c. nb^2 , & resultat formula, quæ nunquam possit esse quadratus: e. g. quantitas $4nx - 1$ nunquam possit esse divisor quantitatis $a^2 + nb^2$, unde $4nx - 1 \times u$ (ubi u sit integer numerus) haud possit æquari $a^2 + nb^2$, & exinde (si modo supponatur $b = 1$) $4nxu - u - n$ haud possit esse quadratus (a^2).

Hic visum est adjicere quasdam alias consimiles propositiones Euleri.

Sit P divisor quantitatis $a^2 + 2b^2$, sed haud sit divisor quantitatum a vel b , tum datur numerus minor quam $\frac{3}{4}P^2 = c^2 + 2d^2$, cujus P sit divisor. Sit etiam P divisor quantitatis $a^2 + b^2$, ubi a & b sint primi inter se, tum P etiam dividet quantitatem $c^2 + d^2$, ita ut $c^2 + d^2$ haud major sit quam $\frac{1}{2}P^2$ & in genere sit P divisor quantitatis $a^2 + nb^2$, ubi P haud sit divisor quantitatum a vel b , & a & b sint inter se primi, tum P erit etiam divisor quantitatis $c^2 + nd^2$, quæ haud major sit quam $\frac{n+1}{4}P^2$.

“ Si formula $a^m - 1$ fuerit divisibilis per primum numerum $p = mn + 1$, tum dantur numeri x & y ejusmodi, ut $ax^m - y^m$ sit per eundem primum numerum p divisibilis.” “ Si etiam hæc forma $ab^n - c^n$ fuerit divisibilis per numerum primum $p = mn + 1$, tum sumpto d pro lubitu dummodo per p haud sit divisibilis, semper inveniri potest numerus x , ut vel hæc forma $ax^n - d^n$ vel $ad^n - x^n$ vel $d^n - ax^n$ vel $x^n - ad^n$ fiat per eundem numerum $p = mn + 1$ divisibilis.”

Sit n primus & a quilibet numerus, & possunt esse $\frac{n+1}{2}$ & haud plura diversa residua e divisione omnium quadratorum a^2 per n , quo-

rum duo semper erunt 0 & 1: e. g. fit $n = 3$ & $\frac{n+1}{2} = 2$ & duo residua quadratorum per 3, erunt 0 & 1: fit $n = 5$ & $\frac{5+1}{2} = 3$, ergo tria sint residua e divisione per 5, quæ erunt 0, 1, 4: quatuor vero erunt residua e divisione per septem, viz. 0, 1, 4, 2, &c. Iisdem positis non possunt esse plura residua biquadratorum a^4 per n quam $\frac{n+3}{4}$, si $\frac{n+3}{4}$ sit integer numerus, sin aliter erunt $\frac{n+1}{2}$ residua.

Sint residua e divisione numerorum $A, B, C, \&c.$ per numeros $a, \beta, \gamma, \&c.$ tum residuum e divisione $\frac{A \times B \times C \times \&c.}{a}$ erit $= a \beta \gamma \&c.$

Sit residuum e divisione a^n per $p = 1$, tum residua ex divisione $a^{2n}, a^{3n}, \&c.$ per p etiam erunt 1.

Sit p primus numerus, & a^n minima potestas quantitatis a , quæ per p divisa relinquit unitatem, & fit m minor quam $p - 1$ & haud major erit quam $\frac{p-1}{2}$; sit minor quam $\frac{p-1}{2}$ & haud major erit quam $\frac{p-1}{3}$; sit minor quam $\frac{p-1}{3}$ & haud major erit quam $\frac{p-1}{4}$: & sic deinceps.

Sit q primus numerus, & a^q per primum numerum p divisus relinquit unitatem, tum a^q erit minima potestas quantitatis a , quæ per p divisa relinquit unitatem, ni a per p divisus relinquit unitatem.

Sit numerus $A = a b c d \&c.$ & sit numerus residuorum e divisione per quantitates $a, b, c, d, \&c.$ respective $m, n, r, s, \&c.$ tum haud plura possint esse residua e divisione per A quam $m n r s \&c.$ frequenter autem quædam ex his fiant bis, ter, &c. inter se æqualia. Hinc non possint esse plura residua e divisione quadratorum per numerum

A quam

A quam $\frac{a-1}{2} \times \frac{b-1}{2} \times \frac{c-1}{2} \times \&c.$ forsan vero haud tam multa inveniuntur.

Data quantitate, quæ sit potestas e pluribus quantitatibus composita, in simplices terminos, eam per binomiale &c. theorema reducendo, haud raro facile constant residua e divisione datæ quantitatis per alteram: e. g. $a^n + 1$ divisa per a relinquit 1: divisa vero per a^2 relinquit $ma + 1$: & sic deinceps.

Omnes potestates a^m , ubi a & m sint integri numeri habent suas figuras desinentes in 0, 1, 5, 6: & omnes quadrati numeri; i. e. a^2 habent suas figuras desinentes in 0, 1, 5, 6, 4, 9: omnes quadrati numeri habent pro suâ penultimâ figurâ 0, 2, 4, 6, 8: ni ultima figura sit 6, in quo casu semper habent imparem numerum: si ultima figura sit 5, tum penultima omnis paris potestatis erit 2, imparis vero vel 2 vel 7.

Visum est subungere propositiones negativas ex aliis petitas:

1. Sint r & s rationales quantitates, & tria latera rationalis trianguli erunt respective $r^2 + s^2$, $r^2 - s^2$ & $2rs$; & ejus area $r \times s \times \frac{r+s}{2} \times \frac{r-s}{2}$. "Hæc vero area non potest esse quadratus, si enim esset quadratus, tum darentur duo quadrato-quadrati, quorum differentia esset quadratus, unde sequitur dari duos quadratos, quorum summa & differentia esset quadratus: datur itaque numerus compositus ex quadrato & duplo quadrati æqualis quadrato, eâ conditione ut quadrati eum componentes faciant quadratum: sed si numerus quadratus componitur ex quadrato & duplo alterius quadrati; ejus latus similiter componitur ex quadrato & duplo quadrati: unde concludetur latus illud esse summam laterum circa rectum trianguli rectanguli & unum ex quadratis illud componentibus efficere Basim & duplum quadratum æquari perpendicularo: illud itaque triangulum conficietur e duobus quadratis, quorum summa & differentia erunt quadrati: at isti duo quadrati minores demonstrabuntur quadratis primo suppositis, quorum tam summa, quam differentia faciunt quadratum; ergo, si dentur duo quadrati, quorum summa & differentia faciunt

faciunt quadratum, dabitur in integris summa duorum quadratorum ejusdem naturæ priore minor; & sic in infinitum, quod impossibile est.

2. Nullus quadratus præter 25 ad numerum 2 adjunctus fiet cubus.
3. Quadrata quatuor haud possunt esse in arithmetica progressionem.
4. Summa duorum cuborum haud æquatur tertio cubo, & sic summa duorum quadrato-quadratorum haud æquatur tertio quadrati quadrato, & sic deinceps: & similiter $a e \times \overline{a^2 - e^2}$ haud possit conficere rationale quadratum, nec $x \times \frac{x+1}{2}$ potest esse quadratus numerus, ni $x = 1$.

Hæ proprietates a Fermatio, Wallisio, &c. inventæ fuerunt: celeberrimus Euler adjecit, & demonstravit subsequencia theoremata.

Nec $\alpha^3 \beta x^4 \pm \beta^3 \alpha y^4$, nec $\alpha^3 \beta x^4 + 2 \beta^3 \alpha y^4$, $2 \alpha^3 \beta x^4 \pm 2 \beta^3 \alpha y^4$, $2 \alpha^3 \beta x^4 - 4 \alpha \beta^3 y^4$, $a e \times \overline{a^2 \pm e^2}$, $2 a e \times \overline{a^2 \pm e^2}$, $x^4 \pm 4 y^4$, nec $4 x^4 - y^4$ possunt esse quadrata: demonstravit etiam si formula $x^4 + k y^4$ quadratum esse non potest, tum $2 k \alpha \beta^3 x^4 - 2 \alpha^3 \beta y^4$ & $\alpha^3 \beta x^4 + k \alpha \beta^3 y^4$, &c. conficere quadratum nequeunt; nec possit rationalis cubus unitate auctus vel diminutus quadratum efficere præter unicum casum, quo cubus est 8, nec $x^6 \pm y^6$ potest esse quadratum, nec numerus triangularis $x \times \frac{x+1}{2}$ præter unitatem potest esse cubus numerus.

D^r. Goldbatch asserit $x \times \overline{\pm 1 + e x + f x^2 + g x^3 + \&c.}$ ubi e, f, g, x , &c. integros numeros respective denotent, haud posse conficere quadrato-quadratum, nec sextæ, decimæ, nec ullius paris potestatis numerum nisi quadratum, præter casum in quo e, f, g , &c. nihilo ponentur æquales: hic etiam adjicere liceat casum, in quo x sit 1; etiamque si x haud sit 1, nec e, f, g , &c. evanescant, tum $x \times \overline{\pm 1 + e x + f x^2 + g x^3 + \&c.}$ nec potest esse quadratus nec cubus nec ulla potestas par vel impar, & magis generaliter factum $a \times x \times \overline{\pm d + e x + f x^2 + g x^3 + \&c.}$ ubi x, a, d, e, f, g , &c. integri sint numeri, non possit esse quadratus, cubus numerus, &c. ni x sit divisor quantitatis a vel d .

Ut

Ut redeam ad Clar. Goldbatch, ipse dicit nec $x \times \overline{x+1} (\tau) + 2$ vel $\tau + 5$ esse quadratum, vel $\tau + 3$ esse numerum quintæ potestatis, vel $\tau + 4$ esse numerum potestatis octavæ vel decimæ.

1. Principia, e quibus demonstrantur præcedentes propositiones ponuntur vel 1^{mo} in perscrutando e quibus generibus necessario constant datæ quantitates, e. g. sit $a^2 + b^2 = c^2$ & pernotum est a, b, c esse quantitates subsequenti generis vel relationis inter se, viz. $a = 2pq$, $b = p^2 - q^2$, $c = p^2 + q^2$, & substituendo quantitates quæsi generis pro datis quantitatibus: 2^{do} correspondentes partes inter se convenire vel æquales esse supponendo, e. g. si numerus requiratur quadratus, supponendo omnes primos divisores inter se quadratos, &c. 3^{io} substituendo pro datis quantitatibus quasdam alias, quæ substitutiones e præcedentibus petendæ sunt: 4^{to} observando aliquam absurdi notam propositionem in se continere, e. g. si talis quadratus numerus requiratur, qui divisus per 3 relinquit 2, quod impossibile est, ergo impossibile est talem quadratum numerum invenire.

P R O B. L.

Datâ æquatione duas vel plures (x, y, &c.) incognitas quantitates habente, & quarum valores sint rationales, invenire æquationem, cujus incognita quantitas (w) quamcunque habeat algebraicam relationem ad radices incognitarum quantitatum datæ æquationis.

1. Si relatio sit algebraica inter quantitatem assumptam (w) & correspondentes valores incognitarum quantitatum, tum prius perficitur problema e reductione plurium æquationum in unam, ita ut incognitæ quantitates exterminentur.

2. Si vero algebraica relatio haud solummodo consistat inter correspondentes rationales radices incognitarum quantitatum, sed involvat diversas rationales radices vel ejusdem incognitæ quantitatis vel rationales haud correspondentes radices e diversis incognitis quantitatibus: Inveniatur relatio inter quascunque diversas radices incognitarum quantitatum, quæ semper inveniri potest, cum detur generalis methodus detegendi unam radicem; methodus enim, quæ generaliter

ter inveniatur unam radicem, etiam plures (si modo plures sint) radices deduceret.

Ita reducantur æquationes resultantes in unam ut exterminentur incognitæ quantitates, & fit problema.

3. Si vero haud inveniri possit relatio inter diversas prædictas radices, tum solummodo e continuis approximationibus, correctionibus & conjecturis progredi liceat.

Et sic de summis e singulis integris valoribus functionum prædictorum generum, ita ut hæ summæ fiant quantitates dati generis. Et de transformandis pluribus hujusce generis æquationibus.

4. Datis quibuscunque quantitatibus vel numeris, ex iis formare dati generis quantitates. Assumantur quædam ex his quantitatibus, & ex iis inveniantur quantitates resultantes: si vero ex his quantitatibus haud formantur quantitates dati generis; continuo corrigantur vel approximantur, usque donec ad veros valores perventum est.

Ex. Sint quantitates datæ omnes cubi numeri, & quantitas dati generis quadratus numerus, & requiratur invenire cubum numerum, qui additus omnibus suis partibus aliquotis conficiat quadratum.

Assumantur multi cubi numeri minimi, & inveniantur aggregata e singulis horum divisoribus, & quando cubi (primi, secundi, tertii, aliive) variorum primorum non habent sua divisorum aggregata ita per primorum paria designanda (qualia habet aggregatum divisorum cubi $3 \ 4 \ 3 = 7^3$: aggregatum enim est $1+7+49+343 = 400 = 20 \times 20 = 2 \times 2 \times 2 \times 2 \times 5 \times 5 = 2^4 \times 5^2$) potest tamen compositus e duobus, tribus, pluribusve talibus continuo multiplicatis (qui itidem cubus erit) habere divisorum suorum aggregatum (quod est compositum ex aggregatis illis continue multiplicatis) sic designandum: nimirum, si cubi sic componendi ita eligantur, ut primi, qui in designandis aggregatorum aliquibus sunt solitarii, comites sibi acquirant ex similibus, qui in aliis aggregatorum occurrunt itidem solitarii; i. e. continuo corrigantur.

Sic pro cubo numeri 47, divisorum aggregatum (numeris primis designatum)

designatum) est $2^2 \cdot 2^2 \cdot 2 \cdot 3 \cdot 5 \cdot 13 \cdot 17$: quod (præter paria) habet solitarios 2, 3, 5, 13, 17; cum quo si componatur cubus numeri 5, qui habet aggregatum divisorum $2^2 \cdot 3 \cdot 13$; hic occurrunt solitarii 3, 13; qui (comites facti prius solitariis 3, 13) relinquunt jam solitarios 2, 5, 17: si tum prius sumpti componantur porro cum cubo numeri 13, qui divisorum aggregatum habet $2^2 \times 5 \times 7 \times 17$; comites habentur prius solitariis 5, 17; sed solitarii supersunt 2, 7: tum assumatur cubus numeri 41, cujus habetur divisorum aggregatum $2^2 \times 3 \times 7 \times 29^2$, & exinde præter paria habentur solitarii 2, 3: & sic pro cubo numeri 11, divisorum aggregatum est $2^2 \times 2 \times 3 \times 61$, unde numeris 2, 3 habentur comites, sed manet 61 solitarius; & pro cubo numeri 27 habetur divisorum aggregatum $2^2 \times 11^2 \times 61$; unde numerus 61 habebit comitem, & radix cubi quæsitæ, ex hac continuâ methodo corrigendi præcedentes assumptos numeros, resultabit $47 \times 5 \times 13 \times 41 \times 11 \times 27$: & sic inveniri possint infinitæ aliæ hujusce problematis solutiones.

Et sic de omnibus hujusce generis problematibus resolvendis.

Hic visum est subjungere duas vel tres subsequentes proprietates primorum numerorum.

Omnis par numerus constat e duobus primis numeris, & omnis impar numerus vel est primus numerus, vel constat e tribus primis numeris, &c.

Haud dantur tres primi numeri in arithmetica progressione, quorum communis differentia haud divisibilis sit per numerum 6; ni 3 sit primus terminus arithmetice seriei, in quo casu possunt esse tres & haud plures termini ejusdem arithmetice seriei primi numeri, & quorum communis differentia haud divisibilis sit per 6: Hic excipiantur duæ arithmetice series 1, 2, 3 & 1, 3, 5, 7.

Haud dantur quinque primi numeri in arithmetica progressione, quorum communis differentia haud divisibilis sit per 30; ni 5 sit primus terminus arithmetice seriei in quo casu possunt esse quinque termini, & haud plures, qui sint primi numeri ejusdem arithmetice seriei: & sic in genere haud dantur 3, 5, 7, 11, 13, 17, &c. primi numeri

meri in arithmetica progressione & quorum communes differentiae haud respective divisibiles sint per numeros $1 \times 2 \times 3$, $1 \times 2 \times 3 \times 5$, $1 \times 2 \times 3 \times 5 \times 7$, $1 \times 2 \times 3 \times 5 \times 7 \times 11$, $1 \times 2 \times 3 \times 5 \times 7 \times 11 \times 13$, $1 \times 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17$, &c. ni 3, 5, 7, 11, 13, 17, &c. sint respective primi termini arithmeticae seriei, in quibus casibus possint esse solummodo 3, 5, 7, 11, 13, 17, &c. termini arithmeticarum serierum respective, quorum differentiae praedictos divisores haud admittant.

Haec proprietas primorum numerorum facile demonstrari possit, & plures ex eodem fonte hauriri possint.

Sit n numerus primus, & $\frac{1 \times 2 \times 3 \times 4 \dots n-2 \times n-1 + 1}{n}$ erit integer numerus, e. g. $\frac{1 \times 2 + 1}{3} = 1$, $\frac{1 \cdot 2 \cdot 3 \cdot 4 + 1}{5} = 5$, $\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 + 1}{7} = 103$, &c. Hanc maxime elegantem primorum numerorum proprietatem invenit vir clarissimus, rerum que mathematicarum peritissimus Joannes Wilson Armiger.

Sed datâ aliquâ quantitate, cujus n sit divisor, facile deduci possint infinitae aliae, quarum eadem quantitas (n) erit divisor: e. g.

$$\frac{1 \cdot 2 \cdot 3 \dots n-2 - 1}{n}, \frac{1 \cdot 2^2 \cdot 3 \dots n-3 + 1}{n}, \frac{1 \cdot 2^2 \cdot 3^2 \dots n-4 - 1}{n}$$

$$\dots \frac{2^1 3^2 4^2 5^2 \dots \frac{n-1}{2}}{n} \pm 1 \quad (\text{ubi erit } + 1, \text{ quando } \frac{n-1}{2} \text{ sit par numerus, sin aliter } - 1) \text{ integri erunt numeri.}$$

Demonstrationes vero hujusmodi propositionum eo magis difficiles erunt, quod nulla fingi potest notatio, quæ primum numerum exprimat.

8 JY 62

SCHOLIUM.

Haftenus de algebraicis æquationibus, eadem etiam applicari possint ad ullas alias æquationes, quæ e substitutione transformari possint

sint in algebraïces æquationes, e. g. sit $x^{xx} + Ax^{x-1} + Bx^{x-2} + \&c. = 0$, pro x^x scribatur z & transformatur data exponentialis æquatio in algebraïcam $z^n + Az^{n-1} + Bz^{n-2} + \&c. = 0$, cujus radices sint $\alpha, \beta, \gamma, \delta, \&c.$ & erit $x^x = \alpha, x^x = \beta, \&c.$ unde (n) erunt diversi valores quantitatis (x^x) .

Sit $x^x = a$, & duæ erunt possibiles radices incognitæ quantitatis x , quarum una erit negativa, altera vero affirmativa, cum a interponatur inter $,69 \&c.$ & $\frac{1}{,69} = 1,44 \&c.$ unus vero solummodo erit valor affirmativus, cum a major sit quam $1,44 \&c.$ unus vero negativus, cum a sit negativus valor & inter negativas quantitates $—,69 \&c.$ & $—1,44 \&c.$ positus, in omni reliquo casu nullus erit possibilis valor.

Sint vero plures exponentiales quantitates in datâ æquatione contentæ, etiamque exponentiales superiorum ordinum; & inveniatur, quot diversos valores habet singula exponentialis diversi generis quantitas: & componantur numeri resultantes, & exinde sæpe inveniri possit numerus valorum incognitæ quantitatis in datâ æquatione: Et sic de transformatione æquationum hujusce generis, de inveniendis impossibilibus, affirmativis & negativis, integris & rationalibus earum radicibus. Sit $x^x = \pm \&c. = a$, ubi a sit integer numerus, & si modo x sit integer numerus erit divisor numeri a , &c. Omnes numeri minime constare possint e finitâ multitudine numerorum x^x vel a^x , ubi a & x integros denotent numeros: successivi enim termini in nimis magnâ ratione crescunt & decrescunt: sic e. g. $a^{x+1} - 1$ semper ad minimum constate pluribus quam $a - 1 \times x$ terminis hujus generis a^x . Sed de exponentialibus scribere non est animus; & hisce meditationibus finis jam imponendus est: hoc contentus, quod transformationes, constitutiones & reductiones æquationum modis ni fallor magis generalibus quam ab aliis factum est, tradiderim.

CORRIGENDA.

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